# Bicategories in Noncommutative Geometry

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CHAPTER 4

# The bimodule bicategory of rings

The 2-category $\mathcal{C}^*(2)$ only has ordinary $^*$-isomorphisms as equivalences. In Chapter 5, we introduce the correspondence bicategory of C*-algebras, whose equivalences are the Morita–Rieffel equivalences. Here we study a simpler, purely algebraic variant of it: the bicategory $\mathfrak{Rings}$, which has rings as objects, bimodules as arrows and bimodule maps as 2-arrows. The equivalences in this bicategory are the Morita equivalences of rings. We also define bicategories and study analogues of covariance algebras in $\mathfrak{Rings}$.

Group rings or enveloping algebras of Lie algebras are defined by a universal property, which says that modules over them are equivalent to representations of some other algebraic structure. The ring is merely a succinct algebraic object to describe a module category. Then it is natural to consider two rings as equivalent when their module categories are equivalent. This is the concept of Morita equivalence, and it is the starting point of this chapter. By definition, this is a 2-categorical concept, formulated in the 2-category of categories, which has categories as objects, functors as arrows, and natural transformations between them as 2-arrows. We may shrink this 2-category a bit. First, we may restrict the objects to module categories of rings. Secondly, we may restrict the arrows to equivalences of categories. Actually, we allow the somewhat larger class of colimit-preserving functors. Any equivalence of categories preserves colimits. A theorem by Eilenberg and Watts says that a functor between module categories that preserves colimits is naturally isomorphic to a functor that tensors with a bimodule. So it seems that we may replace the arrows in our 2-category by the bimodules that induce them. This is, indeed, possible. But the product of bimodules that we get by transferring the composition of functors is only associative and unital up to canonical isomorphisms. Thus rings with bimodules as arrows no longer form a 2-category. Instead, they form a bicategory, which we call $\mathfrak{Rings}$.

The way we arrived at it, the bicategory $\mathfrak{Rings}$ is obviously "equivalent" to the 2-category that has rings as objects, colimit-preserving functors between their module categories as arrows, and natural transformations between these as 2-arrows. This is typical: by MacLane's Coherence Theorem, any small bicategory is "equivalent" to a strict 2-category. The appropriate concept of equivalence here is somewhat technical to write down, however. Equivalence of categories is a 2-categorical concept because it involves categories, functors and natural transformations, which form a 2-category. By analogy, equivalence of bicategories is formulated in the tricategory of bicategories, which has four layers of structure, namely, bicategories as objects, morphisms between bicategories as arrows, transformations between these morphisms as 2-arrows, and modifications between transformations as 3-arrows. The $j$-arrows carry $j$ different products, which are subject to various compatibility axioms, which form the definition of a tricategory. All this would take a while just to write down. What makes it worse is that a bicategory may have non-invertible 2-arrows, and there are variants of morphisms and transformations where the 2-arrows in the data are required to be invertible or not. As a result, this chapter contains quite a few technical definitions.

Since MacLane's Coherence Theorem says that any small bicategory is "equivalent" to a 2-category, why not stay within 2-categories? Our example bicategory $\mathfrak{Rings}$ already answers this question: I expect that most readers will prefer this rather concrete bicategory over the 2-category that has colimit-preserving functors between the module categories as arrows. In addition, bicategories also clarify some of the cohomological computations in Section 2.8 and, in particular, Theorem 2.8.8. In short, any 2-group is equivalent to a "bigroup" that has the extra property that two arrows that are isomorphic through a 2-arrow are already equal. Here a bigroup is a bicategory with one object in which all arrows and 2-arrows are invertible. Equivalence of crossed modules is defined so that it is a special case of equivalence of bigroups. Analysing the definition of a bigroup shows exactly why the MacLane–Whitehead obstruction arises. Actions of the original crossed module are equivalent to actions of the corresponding bigroup, which usually are simpler.

A strictly unital morphism from a group or 2-group to the 2-category $\mathcal{C}^*(2)$ is the same as a twisted action of a group or crossed module on a C*-algebra. Since all 2-arrows in $\mathcal{C}^*(2)$ are invertible, there is no difference between morphisms and homomorphisms here. This suggests to view morphisms or homomorphisms to the bicategory $\mathfrak{Rings}$ as generalised dynamical systems. But should we use morphisms or homomorphisms? In this chapter, we will explore what these concepts and the concepts of transformations, strong transformations, and modifications give for the bicategory $\mathfrak{Rings}$. We will also define different kinds of covariance algebras for them. As it turns out, a strictly unital morphism from a monoid $M$ to $\mathfrak{Rings}$ is equivalent to an $M$-graded ring. When we interpret this as a generalised $M$-action, the graded ring itself is a kind of covariance ring and the subring of elements of degree $1 \in M$ is the ring on which the action takes place; here 1 denotes the unit element in $M$. A strictly unital *homo*morphism $M \to \mathfrak{Rings}$ corresponds to a ring with a *saturated* $M$-grading. The invertible transformations are just graded Morita equivalences, and they become the usual equivariant Morita equivalences when we specialise to actions of $M$ in the usual sense. Strong transformations to a constant diagram play the role of covariant representations.

**Write more about covariance algebras and so on? How much?**

Similar results hold for morphisms defined on categories. And for morphisms defined on 2-categories, we only add some bimodule maps between the homogeneous subspaces of the graded rings for the 2-arrows in the 2-category.

This chapter uses more category theory than the others. This may make it hard for some readers. The concept of a bicategory, and morphisms and transformations in this generality are crucial for later. The particular bicategory $\mathfrak{Rings}$ and our study of Morita equivalence of rings in Section 4.1 are less important for the later chapters. We will observe very similar phenomena in the correspondence bicategory of C*-algebras. For instance, we will also relate graded C*-algebras and strictly unital homomorphisms to the C*-correspondence bicategory; and Morita–Rieffel equivalences of C*-algebras are the same as equivalences in the C*-correspondence bicategory. The bicategory $\mathfrak{Rings}$ has the advantage of not requiring any analysis. Some readers will, therefore, find it an attractive toy model for analogous constructions in the C*-correspondence bicategory.

### 4.1. Morita equivalence for rings

Definition 4.1.1 (Kiiti Morita [**22**]). Let $R$ be a ring. Let $\mathfrak{Mod}_R$ be the category with left $R$-modules as objects, module homomorphisms as arrows, and the usual composition. Two rings $R$ and $S$ are *Morita equivalent* if $\mathfrak{Mod}_R$ and $\mathfrak{Mod}_S$ are equivalent categories.

If a functor $\mathfrak{Mod}_R \to \mathfrak{Mod}_S$ is an equivalence, then it preserves both limits and colimits (see [**23**, Lemma 3.3.6]). We are going to show that functors between module categories that preserve limits or colimits have special forms. This leads to the well known description of Morita equivalence using bimodules.

We begin by recalling the tensor product of Abelian groups and the balanced tensor product of modules over a ring. Since these are preliminary results, we mainly do this in the form of exercises.

DEFINITION 4.1.2. Let $A$ and $B$ be Abelian groups. The *tensor product* $A \otimes B$ is an Abelian group defined follows. It is the quotient of the free Abelian group on the set of symbols $\{a \otimes b : a \in A,\ b \in B\}$ modulo the relations

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b \text{ for all } a_1, a_2 \in A,\ b \in B,$$
$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2 \text{ for all } a \in A,\ b_1, b_2 \in B.$$

EXERCISE 4.1.3. *Let $\mathfrak{Ab}$ be the category of Abelian groups and group homomorphisms.*

(1) *Let $f\colon A \to A'$ and $g\colon B \to B'$ be two homomorphisms of Abelian groups. Show that there is a unique homomorphism $f \otimes g\colon A \otimes B \to A' \otimes B'$ that satisfies*

$$f \otimes g(a \otimes b) = f(a) \otimes f(b) \qquad \text{for all } a \in A,\ b \in B. \tag{4.1.1}$$

*Prove that this makes the construction of $A \otimes B$ a bifunctor $\mathfrak{Ab} \times \mathfrak{Ab} \to \mathfrak{Ab}$.*

(2) *Show that for any Abelian group $A$, the tensor product $A \otimes \mathbb{Z}^n$ is isomophic to $A^n$.*

(3) *Calculate $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$ for natural numbers $n, m \in \mathbb{N}$.*

EXERCISE 4.1.4. *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

*be an exact sequence of Abelian groups. This means that $f$ is injective and that $g$ descends to an isomorphism $B/f(A) \cong C$. Suppose that $D$ is another Abelian group.*

(1) *Prove that the sequence*

$$A \otimes D \xrightarrow{f \otimes \mathrm{id}} B \otimes D \xrightarrow{g \otimes \mathrm{id}} C \to 0$$

*is exact. That is, $g \otimes \mathrm{id}$ induces an isomorphism from $B \otimes D/(f \otimes \mathrm{id})(A \otimes D)$ onto $C \otimes D$.*

(2) *Give an example where $f \otimes \mathrm{id}$ is not injective.*

DEFINITION 4.1.5. Let $R$ and $S$ be two rings, let $Q$ be an $S, R$-bimodule, and $M$ an $R$-module. The *$R$-balanced tensor product* $Q \otimes_R M$ is the quotient of $Q \otimes M$ by the subgroup generated by $q \cdot r \otimes m - q \otimes r \cdot m$ for all $q \in Q$, $r \in R$, $m \in M$. We still write $q \otimes m$ for the image of $q \otimes m \in Q \otimes M$ in $Q \otimes_R M$. The group $Q \otimes_R M$ carries a unique $S$-module structure with $s \cdot (q \otimes m) := (s \cdot q) \otimes m$ all $s \in S$, $q \in Q$, $m \in M$. If $M$ is an $R, T$-module for a third ring $T$, then $Q \otimes_R M$ carries a unique right $T$-module structure with $(q \otimes m) \cdot t := q \otimes (m \cdot t)$ for all $q \in Q$, $m \in M$, $t \in T$. This makes $Q \otimes_R M$ an $S, T$-bimodule.

EXERCISE 4.1.6. *Let $R$ be an associative ring. Let $M$ be a right and $N$ a left $R$-module.*

(1) *For $R$-module maps $f\colon M \to M'$, $g\colon N \to N'$ check that $f \otimes g$ induces a group homomorphism $M \otimes_R N \to M' \otimes_R N'$.*

(2) *Let $Z(R)$ be the centre of $R$. Show that the formula $z \cdot (m \otimes n) = (m \cdot z) \otimes n$ for $z \in Z(R)$, $m \in M$, $n \in N$ defines a $Z(R)$-module structure on $M \otimes_R N$. In particular, if $R$ is a $k$-algebra for some field $k$, then $M \otimes_R N$ is naturally a $k$-vector space.*

(3) *Let* $R = \mathbb{C}[x]$ *and* $M = \mathbb{C}[x]/(x^n)$, $N = \mathbb{C}[x]/(x^m)$ *for* $n, m \geq 1$. *Describe* $M \otimes_R N$ *as an* $R$*-module (since* $R$ *is commutative, we have* $R = Z(R)$*).*

Exercise 4.1.7. *Prove that the multiplication map* $r \otimes m \mapsto r \cdot m$ *defines an isomorphism* $R \otimes_R M \cong M$ *for any left* $R$*-module* $M$. *Similarly,* $n \otimes r \mapsto n \cdot r$ *defines an isomorphism* $N \otimes_R R \cong N$ *for any right* $R$*-module* $N$.

Let $R$ and $S$ be rings and $Q$ an $S, R$-bimodule. Then $Q \otimes_R \sqcup$ defines a functor $Q \otimes_R \sqcup \colon \mathfrak{Mod}_R \to \mathfrak{Mod}_S$. We shall see that a functor preserves colimits if and only if it is of this form for an essentially unique bimodule $Q$. Even more, any such functor has a right adjoint. We describe this adjoint first. It will be useful to characterise which bimodules can occur in a Morita equivalence.

Let $M$ be an $S$-module. Then there is a left action of $R$ on $\mathrm{Hom}_S(Q, M)$ defined by $(r \cdot f)(q) := f(q \cdot r)$ for all $r \in R$, $f \in \mathrm{Hom}_S(Q, M)$, $q \in Q$.

Lemma 4.1.8. *Let* $R$ *and* $S$ *be rings and* $Q$ *an* $S, R$*-bimodule. There are natural isomorphisms*

$$\mathrm{Hom}_S(Q \otimes_R M, N) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(Q, N))$$

*for all* $R$*-modules* $M$ *and* $S$*-modules* $N$*, which are natural in* $M$ *and* $N$.

Proof. The isomorphism sends a map $f \colon Q \otimes_R M \to N$ to the map $M \to \mathrm{Hom}_S(Q, N)$ that maps $m \in M$ to the map $q \mapsto f(q, m)$. Some computations show that this is a well defined isomorphism and natural in $M$ and $N$ (see [**19**, Theorem 3.1]). □

Theorem 4.1.9. *Let* $R$ *and* $S$ *be rings and let* $T \colon \mathfrak{Mod}_R \to \mathfrak{Mod}_S$ *be a functor. The following are equivalent:*

(1) *there is an* $S, R$*-bimodule* $Q$ *such that* $T$ *is naturally isomorphic to the functor* $Q \otimes_R \sqcup$;
(2) $T$ *has a right adjoint functor;*
(3) $T$ *preserves colimits;*
(4) $T$ *preserves direct sums and is right-exact.*

*Let* $T_1, T_2$ *be functors that satisfy this and let* $Q_1$ *and* $Q_2$ *be* $S, R$*-bimodules for them as in* (1). *There is a natural bijection between bimodule maps* $Q_1 \to Q_2$ *and natural transformations* $T_1 \Rightarrow T_2$. *It maps* $f \colon Q_1 \to Q_2$ *to the natural transformation consisting of the maps* $f \otimes_R M \colon Q_2 \otimes_R M \to Q_1 \otimes_R M$.

Proof. Lemma 4.1.8 shows that (1) implies (2). This implies (3) by [**23**, Theorem 4.5.3]. Coproducts and cokernels in module categories are special cases of colimits; the cokernel of $f \colon M \to N$ is the colimit of the coequaliser diagram formed by the pair of maps $f, 0 \colon M \rightrightarrows N$. So a functor that preserves colimits preserves coproducts and cokernels. Coproducts in $\mathfrak{Mod}_R$ are the same as direct sums. A functor $\mathfrak{Mod}_R \to \mathfrak{Mod}_S$ is additive if and only if it preserves finite direct sums (see [**7**, Proposition 1.3.4]). By definition, an additive functor is right-exact if and only if it preserves cokernels (the analogous statement for left-exact functors is [**23**, Proposition 4.5.10]). So (3) implies (4). The main point of the proof is that (4) implies (1).

Any $R$-module $M$ has a free resolution $\bigoplus_{i \in I_1} R \xrightarrow{d} \bigoplus_{i \in I_0} R \twoheadrightarrow M$. A right-exact functor $T$ satisfies $T(M) \cong \operatorname{coker} T(d)$. Using this natural isomorphism, the entire functor – including its action on arrows – is determined by its restriction to the subcategory $\mathfrak{Mod}_R^{\mathrm{free}}$ of free modules. We need the following stronger form of this statement. Let $T \colon \mathfrak{Mod}_R^{\mathrm{free}} \rightrightarrows \mathfrak{Mod}_S$ be a functor. Then $T$ extends naturally to a right-exact functor $\tilde{T} \colon \mathfrak{Mod}_R \rightrightarrows \mathfrak{Mod}_S$, which maps $M$ to $\operatorname{coker} T(d_M)$ for a free resolution as above. As a consequence, if $T_1, T_2 \colon \mathfrak{Mod}_R \to \mathfrak{Mod}_S$ are right-exact functors, then any natural transformation between their restrictions to $\mathfrak{Mod}_R^{\mathrm{free}}$ extends uniquely to a natural transformation $T_1 \Rightarrow T_2$.

To prove the claim above, we need to know in what sense the free resolution above is functorial. Let $M$ and $N$ be two $R$-modules and let $\bigoplus_{i\in I_1} R \xrightarrow{d_M} \bigoplus_{i\in I_0} R \twoheadrightarrow M$ and $\bigoplus_{i\in J_1} R \xrightarrow{d_N} \bigoplus_{i\in J_0} R \twoheadrightarrow N$ be free resolutions. First, any $R$-module map $f\colon M \to N$ is part of a commuting diagram as follows:

$$\begin{array}{ccccc}
\bigoplus_{i\in I_1} R & \xrightarrow{d_M} & \bigoplus_{i\in I_0} R & \twoheadrightarrow & M \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\
\bigoplus_{i\in J_1} R & \xrightarrow{d_M} & \bigoplus_{i\in J_0} R & \twoheadrightarrow & N.
\end{array}$$

Secondly, if $f_1, f_0$ and $f_1', f_0'$ are two ways to make this diagram commute, then there is an $R$-module map $h\colon \bigoplus_{i\in I_0} R \to \bigoplus_{i\in J_1} R$ with $f_0' - f_0 = d_N \circ h$. These two elementary statements follow from basic results in homological algebra, and we omit the proof.

A pair of maps $(f_0, f_1)$ as above induces a map $(f_0, f_1)_*\colon \operatorname{coker} T(d_M) \to \operatorname{coker} T(d_N)$ by the naturality of cokernels, and two such pairs with $f_0' - f_0 = d_N \circ h$ satisfy $(f_0, f_1)_* = (f_0', f_1')_*$. This shows that the map sending $M$ to $\operatorname{coker} T(d_M)$ is a functor. We omit the proof that this functor is right-exact. The construction shows that it is the only possible right-exact extension of $T$ to all of $\mathfrak{Mod}_R$.

If the functor $T$ also preserves direct sums, then there are natural isomorphisms $T\big(\bigoplus_{i\in I} R\big) \cong \bigoplus_{i\in I} T(R)$ for all sets $I$. Then the restriction of $T$ to the single module $R$ determines the restriction of $T$ to free modules – including the action on arrows. More precisely, a given functor $T\colon \mathfrak{Mod}_R|_{\{R\}} \to \mathfrak{Mod}_S$ extends naturally and uniquely to a functor $\tilde{T}\colon \mathfrak{Mod}_R^{\mathrm{free}} \to \mathfrak{Mod}_S$ that commutes with direct sums, and any natural transformation between functors $T_1, T_2\colon \mathfrak{Mod}_R|_{\{R\}} \rightrightarrows \mathfrak{Mod}_S$ extends uniquely to a natural transformation $\tilde{T}_1 \Rightarrow \tilde{T}_2$. The extension $\tilde{T}$ of $T$ is defined simply by $\tilde{T}\big(\bigoplus_I R\big) := \bigoplus_I T(R)$.

As a consequence, if two functors $T_1, T_2$ are right-exact and preserve direct sums, then any natural transformation between their restrictions to the full subcategory with only $R$ as an object extends uniquely to a natural transformation $T_1 \Rightarrow T_2$.

Let $Q := T(R)$. This is some left $S$-module. Right multiplication with $r \in R$ is a left module homomorphism $R \to R$, $x \mapsto x \cdot r$. Since $T$ is a functor, this induces a left $S$-module homomorphism on $Q$, which we denote multiplicatively. The distributive law $x \cdot (r_1 + r_2) = x \cdot r_1 + x \cdot r_2$ holds because $T$ is additive (since it preserves finite direct sums). The functoriality of $T$ implies $(x \cdot r_1) \cdot r_2 = x \cdot (r_1 \cdot r_2)$. Thus $Q$ becomes an $S, R$-bimodule.

The multiplication map $Q \otimes_R R \to Q$ is an isomorphism between the restrictions of the functors $Q \otimes_R \sqcup$ and $T$ to the full subcategory with only $R$ as an object. It is natural because it is a right $R$-module homomorphism and $\operatorname{Hom}_R(R, R) \cong R$. We have already shown that (1) implies (4). So both functors $Q \otimes_R \sqcup$ and $T$ preserve direct sums and are right-exact. Hence the natural isomorphism on the single module $R$ extends uniquely to a natural isomorphism between $Q \otimes_R \sqcup$ and $T$ on all $R$-modules.

The proof also gives a natural bijection between natural transformations $T_1 \Rightarrow T_2$ and $S, R$-bimodule homomorphisms $f\colon Q_1 \to Q_2$ for the bimodules $Q_j := T_j(R)$ for $j = 1, 2$. The maps $f \otimes_R \mathrm{id}_M$ for all $R$-modules $M$ form a natural transformation $Q_1 \otimes_R \sqcup \Rightarrow Q_2 \otimes_R \sqcup$ that restricts to $f$ for $M = R$. Hence this is the unique natural transformation extending $f$. □

THEOREM 4.1.10. *Let $R$ and $S$ be rings and let $T\colon \mathfrak{Mod}_R \to \mathfrak{Mod}_S$ be a functor. The following are equivalent:*

(1) *there is an* $R,S$*-bimodule* $Q$ *such that* $T$ *is naturally isomorphic to the functor* $\mathrm{Hom}_R(Q,\sqcup)$*;*
(2) $T$ *has a left adjoint functor;*
(3) $T$ *preserves limits;*
(4) $T$ *preserves direct products and is left-exact.*

*Let* $T_1, T_2$ *be functors that satisfy this and let* $Q_1$ *and* $Q_2$ *be* $R,S$*-bimodules for them as in* (1)*. There is a natural bijection between bimodule maps* $Q_2 \to Q_1$ *and natural transformations* $T_1 \Rightarrow T_2$*. It maps* $f\colon Q_2 \to Q_1$ *to the natural transformation consisting of the maps* $f^*\colon \mathrm{Hom}_R(Q_2, M) \to \mathrm{Hom}_R(Q_1, M)$*,* $h \mapsto h \circ f$*.*

PROOF. The implications (1)$\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(4) are shown as in the proof of Theorem 4.1.9. Now we show that each of these implications may be reversed.

First we show that (2) implies (1). The right adjoint $G$ of a functor $F$ is unique up to natural isomorphism if it exists. Even more, the adjunction induces a natural bijection between natural transformations $F_1 \Rightarrow F_2$ and $G_2 \Rightarrow G_1$ if $G_j$ is right adjoint to $F_j$. Theorem 4.1.9 concretely describes all functors with a right adjoint. Thus any functor with a left adjoint is naturally isomorphic to the right adjoint of $Q \otimes_S \sqcup$ for some $R,S$-bimodule $Q$. Then it is naturally isomorphic to $\mathrm{Hom}_R(Q,\sqcup)$ by Lemma 4.1.8. And natural transformations correspond to bimodule maps as asserted. This finishes the proof that (2) implies (1).

The implication (3)$\Longrightarrow$(2) follows from the Special Adjoint Functor Theorem (see [**23**, Theorem 4.6.10]). To apply it, we need to know that the category $\mathfrak{Mod}_R$ has two properties: every class of subobjects of a fixed object should have an intersection, and there should be a coseparator. Subobjects of a given $R$-module are equivalent to submodules, and the relevant concept of "intersection" becomes the obvious intersection of submodules. Hence any class of submodules of a given module has an intersection, no matter how large the class is. To build a coseparator for $\mathfrak{Mod}_R$, choose an injective resolution $R/J \to I_J$ for each left ideal $J$ in $R$. We claim that their product is a coseparator for $\mathfrak{Mod}_R$. To see this, take two arrows $f, g\colon M \rightrightarrows N$ with $f \neq g$. Then there is $m \in M$ with $f(m) \neq g(m)$. Let $J \subseteq R$ be the annihilator of $(f-g)(m)$. Then $R/J \to N$, $r \mapsto r \cdot (f-g)(m)$, is an injective module homomorphism. The inclusion $R/J \hookrightarrow I_J$ extends to a module homomorphism $N \to I_J$ because $I_J$ is injective. Together with the zero map in the other factors, this gives a map $N \to \prod I_J$. This map does not annihilate $(f-g)(m)$ because $R/J$ embeds into $I_J$. Hence we have indeed got a coseparator.

Finally, (4) implies (3) because the limit of any small diagram may be computed using products and equalisers (see [**23**, Theorem 3.4.12]); and equalisers are equivalent to kernels in $\mathfrak{Mod}_R$. □

Theorem 4.1.9 and Theorem 4.1.10 were discovered simultaneously by Eilenberg, Gabriel, and Watts [**26**] around 1960.

EXAMPLE 4.1.11. A ring homomorphism $f\colon S \to R$ induces a functor

$$f^*\colon \mathfrak{Mod}_R \to \mathfrak{Mod}_S,$$

which maps an $R$-module $M$ to the same Abelian group with the $S$-module structure $s \cdot m := f(s) \cdot m$. The functor $f^*$ is exact and preserves direct sums and products because it does not change the underlying Abelian group. By Theorem 4.1.9, it must be of the form $Q \otimes_R \sqcup$ for some $S,R$-bimodule $Q$. The proof of the theorem shows that $Q$ is $R$ as a right $R$-module, with the left $S$-module structure $s \cdot r := f(s)r$ for all $s \in S$, $r \in R$. In particular, the identity functor on $\mathfrak{Mod}_R$ corresponds to $R$ with the obvious $R$-bimodule structure.

THEOREM 4.1.12. *Two rings $R$ and $S$ are Morita equivalent if and only if there are an $S,R$-bimodule $Q$ and an $R,S$-bimodule $P$ with bimodule isomorphisms $Q \otimes_R P \cong S$ and $P \otimes_S Q \cong R$.*

PROOF. By Theorem 4.1.9, an equivalence of categories $\mathfrak{Mod}_R \xrightarrow{\simeq} \mathfrak{Mod}_S$ and its inverse are of the form $Q \otimes_R \sqcup$ and $P \otimes_S \sqcup$ for an $S,R$-bimodule $Q$ and an $R,S$-bimodule $P$. Since these functors are inverse to each other up to natural isomorphisms, the functors $Q \otimes_R (P \otimes_S \sqcup)$ and $P \otimes_S (Q \otimes_R \sqcup)$ are naturally isomorphic to the identity functors. There are obvious natural isomorphisms $Q \otimes_R (P \otimes_S \sqcup) \cong (Q \otimes_R P) \otimes_S \sqcup$ and $P \otimes_S (Q \otimes_R \sqcup) \cong (P \otimes_S Q) \otimes_R \sqcup$. So these composite functors come from the bimodules $Q \otimes_R P$ and $P \otimes_S Q$, respectively. The identity functors come from the bimodules $R$ and $S$, respectively, by Example 4.1.11. By Theorem 4.1.9, the natural isomorphisms between our tensor product functors are equivalent to bimodule isomorphisms $Q \otimes_R P \cong S$ and $P \otimes_S Q \cong R$. □

The bimodules that can occur in an equivalence are more special because an equivalence of categories preserves both colimits and limits:

THEOREM 4.1.13. *Let $R, S$ be rings. Let $Q$ be an $S,R$-bimodule. The functor $Q \otimes_R \sqcup$ preserves limits if and only if $Q$ is finitely generated and projective as a right $R$-module.*

PROOF. By definition, the right $R$-module $Q$ is projective if and only if it is a direct summand of a free right $R$-module. It is finitely generated and projective if and only if $Q \cong p \cdot R^n$ as a right $R$-module for some $n \in \mathbb{N}$ and some idempotent $p \in \mathbb{M}_n(R)$. Then $Q \otimes_R M \cong p \cdot M^n$. It is easy to see that this functor is exact and preserves products. Then $Q \otimes_R \sqcup$ preserves limits by Theorem 4.1.10.

The functor $Q \otimes_R \sqcup$ is right-exact, anyway. So it is left-exact if and only if it is exact. By definition, this says that the bimodule $Q$ is flat. Lazard showed that any flat module is an inductive limit of finitely generated, projective modules. Then a finitely presented, flat module is projective by [**17**, Corollaire 1.4]. Thus it suffices to prove that $Q$ is finitely presented if $Q \otimes_R \sqcup$ preserves limits.

Since $Q \otimes_R \prod_{q \in Q} R \cong \prod_{q \in Q} Q$, there must be an element of $Q \otimes_R \prod_{q \in Q} R$ that is mapped to the diagonal element $(q)_{q \in Q} \in \prod_{q \in Q} Q$. This element is a finite sum $\sum_{i=1}^n q_i \otimes m_i$ with $q_i \in Q$, $m_i \in \prod_{q \in Q} R$. Then $q_1, \dots, q_n$ must generate $Q$ as an $R$-module; so $Q$ is finitely generated. Hence there is an extension $Q' \rightarrowtail R^n \twoheadrightarrow Q$ for some $n \in \mathbb{N}$ and some $R$-module $Q'$. Both functors $Q \otimes_R \sqcup$ and $R^n \otimes_R \sqcup$ preserve kernels and products; this is inherited by $Q' \otimes_R \sqcup$. Hence the argument above applies to $Q'$ and shows that it is finitely generated. This gives a finite presentation for $Q$. □

Let $Q$ be a finitely generated, projective right $R$-module. Then

$$S := \mathrm{Hom}_R(Q, Q)$$

is a ring under composition of right $R$-module maps, and $Q$ is an $S,R$-bimodule in an obvious way. Identify $R$ with the ring $\mathrm{Hom}_R(R, R)$ of right $R$-module homomorphisms $R \to R$. Then

$$P := \mathrm{Hom}_R(Q, R)$$

is an $R,S$-bimodule through composition of maps. A similar identification $Q \cong \mathrm{Hom}_R(R, Q)$ allows to define bilinear maps

$$P \otimes_S Q \to R, \qquad Q \otimes_R P \to S,$$

by composing right $R$-module maps $R \to Q \to R$ and $Q \to R \to Q$. In order for the above data to define a Morita equivalence $\mathfrak{Mod}_R \cong \mathfrak{Mod}_S$, we need the maps $P \otimes_S Q \to R$ and $Q \otimes_R P \to S$ above to be bijective.

LEMMA 4.1.14. *The map $Q \otimes_R P \to S$ is always bijective. The map $P \otimes_S Q \to R$ is bijective if and only if $R$ is a direct summand in $P^n$ for some $n \in \mathbb{N}$.*

PROOF. Since $Q$ is finitely generated, projective, there are $m \in \mathbb{N}$ and an idempotent element $p \in \mathbb{M}_m(R)$ such that $Q$ is isomorphic to $p \cdot R^m$ with the obvious right module structure. We may replace $Q$ by this isomorphic module. Then we may identify $S \cong p \cdot \mathbb{M}_m(R) \cdot p$ with the multiplication inherited from $\mathbb{M}_m(R)$, and $P \cong R^m \cdot p$. The $S$-module structures on $Q$ and $R$ are given by the multiplication of matrices with column or row vectors, respectively. Then $P \otimes_R Q \cong p \cdot R^m \otimes_R R^m \cdot p \cong p \cdot \mathbb{M}_m(R) \cdot p \cong S$. Here we use many copies of the natural isomorphism $R \otimes_R R \cong R$ from Exercise 4.1.7 to identify $R^m \otimes_R R^m \cong \mathbb{M}_m(R)$.

Now assume that $R$ is a direct summand in $Q^n$. That is, $R \cong q \cdot Q^n$ for some idempotent $R$-module homomorphism $q\colon Q^n \to Q^n$. Since $S = \mathrm{Hom}_R(Q, Q)$, we may view $q$ as an idempotent element in $\mathbb{M}_n(S)$. And there are isomorphisms

$$R \cong \mathrm{Hom}_R(R, R) \cong q \cdot \mathrm{Hom}_R(Q^n, Q^n) \cdot q \cong q \cdot \mathbb{M}_n(S) \cdot q,$$
$$P \cong \mathrm{Hom}_R(Q, R) \cong q \cdot \mathrm{Hom}_R(Q, Q^n) \cong q \cdot S^n.$$

Now the computation above gives $P \otimes_S Q \cong q \cdot \mathbb{M}_n(S) \cdot q \cong R$. □

DEFINITION 4.1.15. A projective $R$-module $P$ is called a *generator* if $R$ is a direct summand in $\bigoplus_{i \in I} P$ for some set $I$.

THEOREM 4.1.16. *A right $R$-module $Q$ is part of a Morita equivalence if and only if it is a finitely generated, projective generator. In this case, the Morita equivalence is between $R$ and the ring $S := \mathrm{Hom}_R(Q, Q)$, with the obvious $S, R$-bimodule structure on $Q$, and involves $P := \mathrm{Hom}_R(Q, R)$ with the $R, S$-bimodule structure by composition of right $R$-module maps.*

PROOF. Since $R$ is projective, it is a direct summand in $\bigoplus_{i \in I} P$ if and only if it is a quotient of $\bigoplus_{i \in I} P$. Since $R$ is finitely generated, we may then replace $I$ by a finite subset. Therefore, $P$ is a projective generator if and only if $R$ is a direct summand of $P^n$ for some $n \in \mathbb{N}$. Now Lemma 4.1.14 and the discussion above it show that any finitely generated, projective generator $Q$ is part of a Morita equivalence. Conversely, assume that $Q$ is part of a Morita equivalence. Then $Q$ is finitely generated and projective by Theorem 4.1.13. By assumption, there are a ring $S$ and an $R, S$-bimodule $P$ so that $Q$ is an $S, R$-bimodule and $P \otimes_S Q \cong R$. Theorem 4.1.13 also applies to $P$ and shows that it is finitely generated and projective as an $S$-module. So $P \cong p \cdot S^n$ for some idempotent $p \in \mathbb{M}_n(S)$. Then $P \otimes_S Q \cong p \cdot S^n \otimes_S Q$ is a direct summand in $S^n \otimes_S Q \cong Q^n$. This says that $Q$ is a generator.

It still remains to prove that the only Morita equivalence that contains $Q$ is the one described above, with $S \cong \mathrm{Hom}_R(Q, Q)$ and $P \cong \mathrm{Hom}_R(Q, R)$. Let $Q'$ be $Q$ viewed as a $\mathbb{Z}, R$-bimodule. Then elements of $\mathrm{Hom}_R(Q, Q)$ are in natural bijection with natural transformations $Q' \otimes_R \sqcup \to Q' \otimes \sqcup_R$ by Theorem 4.1.9. We may rewrite this functor as the composite of the equivalence $Q \otimes_R \sqcup$ and the forgetful functor $\mathfrak{Mod}_S \to \mathfrak{Mod}_{\mathbb{Z}}$. So the natural transformations $Q' \otimes_R \sqcup \to Q' \otimes \sqcup_R$ are in natural bijection with natural transformations from the forgetful functor $\mathfrak{Mod}_S \to \mathfrak{Mod}_{\mathbb{Z}}$ to itself. The forgetful functor is naturally isomorphic to the tensor product with $S$ viewed as a $\mathbb{Z}, S$-bimodule. Therefore, the natural transformations above are in natural bijection with $S$ by Theorem 4.1.9. Thus the ring $S$ is isomorphic to $\mathrm{Hom}_R(Q, Q)$. The other bimodule $P$ in a Morita equivalence is identified with $\mathrm{Hom}_R(Q, R)$ by a similar analysis, starting with natural transformations $Q' \otimes_R \sqcup \Rightarrow R' \otimes_R \sqcup \cong (P' \otimes_S Q) \otimes_R \sqcup$, where $R'$ and $P'$ are the $\mathbb{Z}, R$- and $\mathbb{Z}, S$-bimodules that we get by forgetting the left module structures on $R$ and $P$, respectively. □

EXAMPLE 4.1.17. Let $R$ be a ring and let $n \in \mathbb{N}_{\geq 2}$. Then $R$ is Morita equivalent to $\mathbb{M}_n(R)$. To see this, let $Q := R^n$ as a right $R$-module. This is a finitely generated, projective generator. So it produces a Morita equivalence between $R$ and the ring $\mathrm{Hom}_R(Q, Q) \cong \mathbb{M}_n(R)$ by Theorem 4.1.16.

More generally, the proof of Theorem 4.1.16 shows that any ring Morita equivalent to a given ring $R$ is of the form $p\mathbb{M}_n(R)p$ for some $n \in \mathbb{N}$ and some idempotent element $p \in \mathbb{M}_n(R)$ with the extra property that the corresponding finitely generated, projective module $pR^n$ is a generator.

EXERCISE 4.1.18. *The finitely generated, projective module $pR^n$ is a generator if and only if the two-sided ideal in $\mathbb{M}_n(R)$ generated by $p$ is all of $\mathbb{M}_n(R)$.*

A subalgebra of the form $p\mathbb{M}_n(R)p$ is also called a *corner* in $\mathbb{M}_n(R)$, and the corner and the idempotent $p$ are called *full* if the two-sided in $\mathbb{M}_n(R)$ generated by $p$ is all of $\mathbb{M}_n(R)$. Thus a ring is Morita equivalent to $R$ if and only if it is isomorphic to a full corner in a matrix algebra over $R$. The following exercise gives a more symmetric and "natural" statement of a similar nature:

EXERCISE 4.1.19. *Let $R$, $S$, $P$ and $Q$ be as in Theorem 4.1.12. Show that the isomorphisms $\varphi\colon P \otimes_S Q \xrightarrow{\simeq} R$ and $\psi\colon Q \otimes_R P \xrightarrow{\simeq} S$ can be chosen such that the following diagrams commute:*

$$\begin{array}{ccc} P \otimes_S Q \otimes_R P & \xrightarrow[\cong]{\varphi \otimes_R \mathrm{id}_P} & R \otimes_R P \\ {\scriptstyle \mathrm{id}_P \otimes_S \psi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow \\ P \otimes_S S & \xrightarrow[\cong]{} & P \end{array} \qquad \begin{array}{ccc} Q \otimes_R P \otimes_S Q & \xrightarrow[\cong]{\psi \otimes_S \mathrm{id}_Q} & S \otimes_S Q \\ {\scriptstyle \mathrm{id}_Q \otimes_R \varphi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow \\ Q \otimes_R R & \xrightarrow[\cong]{} & Q \end{array}$$

*Here the unlabelled arrows are the canonical isomorphisms. (Hint: any equivalence of categories may be improved to an adjoint equivalence.)*

*Assume these commuting diagrams. Show that the ring structures on $R$ and $S$, the bimodule structures on $P$ and $Q$, and the maps $\varphi$ and $\psi$ together give an associative ring structure on $L := R \oplus P \oplus Q \oplus S$. This is called the* linking ring *of the Morita equivalence.*

*The element $p := (1, 0, 0, 0)$ in $L$ is idempotent and $p^\perp := 1 - p = (0, 0, 0, 1)$. The subalgebras $pLp$ and $p^\perp L p^\perp$ of $L$ are isomorphic to $R$ and $S$, respectively. The two-sided ideals $(p)$ and $(p^\perp)$ in $L$ that are generated by $p$ and $p^\perp$ are both equal to $L$. (Briefly, $p$ and $p^\perp$ are* full *idempotents in $L$ and $pLp$ and $p^\perp L p^\perp$ are* full, complementary corners *in $L$.)*

EXERCISE 4.1.20. *Two rings $R$ and $S$ are Morita equivalent if and only if their opposite rings $R^{\mathrm{op}}$ and $S^{\mathrm{op}}$ are Morita equivalent.*

Since right $R$-modules are the same as left $R^{\mathrm{op}}$-modules, the exercise above shows that it makes no difference to use right instead of left modules to define Morita equivalence.

EXERCISE 4.1.21 (The centre is Morita invariant).

(1) *Let $R$ be a ring and let $E$ be an $R$-module. Denote by $Z(R)$ the centre of $R$. For $c \in Z(R)$ define a map $m_c\colon E \to E$ as $m_c(e) = c \cdot e$ for all $e \in E$. Show that $m_c$ is an $R$-module homomorphism and $c \mapsto m_c$ is a ring homorphism $Z(R) \to \mathrm{End}_R(E)$.*

(2) *Suppose that rings $R$ and $Q$ are Morita equivalent. Show that $Z(R) \cong Z(Q)$. Conclude that commutative rings are Morita equivalent if and only if they are isomorphic.*

EXERCISE 4.1.22. *Let $R = \mathbb{Z}$ or a field $\mathbb{K}$. Prove that if a ring $Q$ is Morita equivalent to $R$ then $Q \cong \mathrm{M}_n(R)$ for some $n \in Q$.*

EXERCISE 4.1.23. *Let $G$ and $H$ be finite groups. Show that $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are Morita equivalent if and only if $G$ and $H$ have the same number of conjugacy classes. You may use that every representation of a finite group over $\mathbb{C}$ is a direct sum of irreducible finite-dimensional representations and the number of isomorphism classes of irreducible representations is the same as the number of conjugacy classes.*

EXERCISE 4.1.24. *Let $U \subset \mathbb{M}_2(\mathbb{C})$ be the subalgebra of upper triangular matrices. Let $e_1, e_2$ be the canonical basis of $\mathbb{C}^2$. Then $P_1 = \langle e_1 \rangle \subseteq \mathbb{C}^2$ and $P_2 = \langle e_1, e_2 \rangle = \mathbb{C}^2$ with matrix multiplication are left $U$-modules.*

(1) *Show that $P_1 \oplus P_2$ is the free $U$-module of rank 1. Deduce that $P_1$ and $P_2$ are projective and finitely generated.*
(2) *Is $P_1$ or $P_2$ a projective generator?*
(3) *Prove that $P = P_1^2 \oplus P_2$ is a projective generator and explicitly describe $U' = \mathrm{End}_U(P)$ as a subalgebra of $\mathrm{M}_4(\mathbb{C})$.*
(4) *Show that the algebras $U$ and $U'$ are Morita equivalent but $U'$ is not isomorphic to $\mathbb{M}_n(U)$ for any $n \in \mathbb{N}$.*

## 4.2. The bicategory of rings and bimodules

Categories form a 2-category $\mathfrak{Cat}$ by Example 2.2.3. Then we may turn rings into a 2-category by taking all functors $\mathfrak{Mod}_R \to \mathfrak{Mod}_S$ as arrows $R \to S$ for two rings $R$ and $S$. This 2-category, however, has far too many arrows. Theorem 4.1.9 suggests to restrict attention to those functors that preserve colimits or, equivalently, are naturally isomorphic to $Q \otimes_R \sqcup$ for an $S, R$-bimodule $Q$. Instead, we may also take functors that preserve limits, or functors that preserve both limits and colimits. In the latter case, we only get the functors $Q \otimes_R \sqcup$ for $S, R$-bimodules $Q$ that are finitely generated and projective as right $R$-modules (see Theorem 4.1.13). We still take all natural transformations because they all have a simple form by Theorem 4.1.9 and Theorem 4.1.10. The three choices of arrows above define 2-subcategories of $\mathfrak{Cat}$.

A colimit-preserving functor between module categories is "equivalent" to something much more concrete, namely, a bimodule. It should be possible to replace the functor by the corresponding bimodule. Since this bimodule is only unique up to canonical isomorphism, we cannot expect to get a 2-category any more: various identities of arrows in the definition of a 2-category are weakened to invertible 2-arrows. Instead, we only get a "bicategory".

More explicitly, let $R$, $S$ and $T$ be rings, let $Q$ be an $S, R$-bimodule and $P$ a $T, S$-bimodule. Then there are natural isomorphisms

$$(P \otimes_S Q) \otimes_R M \cong P \otimes_S (Q \otimes_R M), \qquad p \otimes (q \otimes m) \mapsto (p \otimes q) \otimes m,$$

for all $R$-modules $M$. Thus the composite functor $P \otimes_S (Q \otimes_R \sqcup)$ is naturally equivalent – but not equal – to the functor associated to the bimodule $P \otimes_S Q$. (This observation was already used in the proof of Theorem 4.1.12.) So we call $P \otimes_S Q$ the *product* of $P$ and $Q$.

Let $P_1, P_2$ and $Q_1, Q_2$ be $T, S$-bimodules and $S, R$-bimodules, respectively. Bimodule homomorphisms $f\colon P_1 \to P_2$ and $g\colon Q_1 \to Q_2$ correspond to natural transformations $f \otimes_R \sqcup\colon P_1 \otimes_S \sqcup \Rightarrow P_2 \otimes_S \sqcup$ and $g \otimes_R \sqcup\colon Q_1 \otimes_R \sqcup \Rightarrow Q_2 \otimes_R \sqcup$. Their horizontal product in $\mathfrak{Cat}$ is a natural transformation $P_1 \otimes_R (Q_1 \otimes_R \sqcup) \Rightarrow P_2 \otimes_R (Q_2 \otimes_R \sqcup)$. When we identify $P_j \otimes_R (Q_j \otimes_R \sqcup)$ with $(P_j \otimes_R Q_j) \otimes_R \sqcup$ for $j = 1, 2$, then this horizontal product becomes equal to the natural transformation associated to the bimodule homomorphism $f \otimes_S g\colon P_1 \otimes_S Q_1 \to P_2 \otimes_S Q_2$. Thus it is reasonable to call $f \otimes_S g$ the *horizontal product* of $f$ and $g$.

Example 4.1.11 shows that the identity functor on the category of $R$-modules is equivalent to the functor $R \otimes_R \sqcup$, where we equip $R$ with the obvious $R$-bimodule structure. Once again, this is only a natural isomorphism of functors, not an equality.

The bimodules $S \otimes_S Q$, $Q$ and $Q \otimes_R R$ for an $S,R$-bimodule $Q$ are canonically isomorphic by Exercise 4.1.7 – but not equal. And if $O$, $P$, $Q$ are bimodules over the pairs of rings $U,T$, $T,S$ and $S,R$, then there is a natural $U,R$-bimodule isomorphism

$$(O \otimes_T P) \otimes_S Q \xrightarrow{\cong} O \otimes_T (P \otimes_S Q), \qquad o \otimes (p \otimes q) \mapsto (o \otimes p) \otimes q.$$

Hence the product of bimodules is unital and associative up to certain natural isomorphisms. As in our discussion of twisted group actions, we have to be careful when replacing identities of arrows by equivalences: the 2-arrows in these equivalences must become part of our data, and they must satisfy suitable coherence conditions. These are contained in the definition of a bicategory (see [**6**, **18**]):

Definition 4.2.1. A *bicategory* is given by the following data:

(1) a set $\mathcal{C}^0$ of objects;
(2) for all objects $x, y \in \mathcal{C}^0$, a category $\mathcal{C}(x,y)$, whose objects are the arrows $x \to y$ of the bicategory and whose arrows are the 2-arrows between these arrows; the category structure provides an associative *vertical product* $\cdot$ on 2-arrows and a *unit 2-arrow* $1_f$ on each arrow $f$;
(3) for all $x, y, z \in \mathcal{C}^0$, a bifunctor $\circ\colon \mathcal{C}(y,z) \times \mathcal{C}(x,y) \to \mathcal{C}(x,z)$; this bifunctor contains a *product* $\circ$ on arrows and a *horizontal product* $\bullet$ on 2-arrows; bifunctoriality says that $1_f \bullet 1_g = 1_{f \circ g}$ for composable arrows $f$ and $g$ and that $\bullet$ commutes with vertical products (see the diagram in Exercise 2.2.1).
(4) for each object $x$, a *unit arrow* $1_x\colon x \to x$;
(5) invertible natural transformations

$$l_f\colon 1_y \circ f \Rightarrow f \quad \text{and} \quad r_f\colon f \circ 1_x \Rightarrow f$$

– called *uniters* – for all arrows $f \in \mathcal{C}(x,y)$;
(6) invertible natural transformations

$$\mathrm{ass}\colon (f_1 \circ f_2) \circ f_3 \Rightarrow f_1 \circ (f_2 \circ f_3)$$

– called *associators* – for all composable arrows $x_0 \xrightarrow{f_3} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_1} x_3$.

For the naturality in the last two conditions, we view $f \mapsto f$, $f \mapsto 1_y \circ f$, and $f \mapsto f \circ 1_x$ as functors $\mathcal{C}(x,y) \to \mathcal{C}(x,y)$ and $(f_1,f_2,f_3) \mapsto (f_1 \circ f_2) \circ f_3$ and $(f_1,f_2,f_3) \mapsto f_1 \circ (f_2 \circ f_3)$ as functors $\mathcal{C}(x_2,x_3) \times \mathcal{C}(x_1,x_2) \times \mathcal{C}(x_0,x_1) \to \mathcal{C}(x_0,x_3)$. Thus the naturality of $l_f$ and $r_f$ say that for any 2-arrow $c\colon f_1 \Rightarrow f_2$ in $\mathcal{C}$ for arrows $f_1, f_2\colon x \rightrightarrows y$, the following diagrams of 2-arrows commute:

$$\begin{array}{ccc} 1_y \circ f_1 & \xRightarrow{l_{f_1}} & f_1 \\ {\scriptstyle 1_y \bullet c}\Downarrow & & \Downarrow{\scriptstyle c} \\ 1_y \circ f_2 & \xRightarrow{l_{f_2}} & f_2 \end{array} \qquad \begin{array}{ccc} f_1 \circ 1_y & \xRightarrow{r_{f_1}} & f_1 \\ {\scriptstyle c \bullet 1_y}\Downarrow & & \Downarrow{\scriptstyle c} \\ f_2 \circ 1_y & \xRightarrow{r_{f_2}} & f_2 \end{array} \tag{4.2.1}$$

The naturality of the associators says that if $f_1, f_1' \in \mathcal{C}(x_2,x_3)$, $f_2, f_2' \in \mathcal{C}(x_1,x_2)$ and $f_3, f_3' \in \mathcal{C}(x_0,x_1)$ and $c_j\colon f_j \Rightarrow f_j'$ are 2-arrows, then the following diagram of 2-arrows commutes:

$$\begin{array}{ccc} (f_1 \circ f_2) \circ f_3 & \xRightarrow{\mathrm{ass}_{f_1,f_2,f_3}} & f_1 \circ (f_2 \circ f_3) \\ {\scriptstyle (c_1 \bullet c_2) \bullet c_3}\Downarrow & & \Downarrow{\scriptstyle c_1 \bullet (c_2 \bullet c_3)} \\ (f_1' \circ f_2') \circ f_3' & \xRightarrow[\mathrm{ass}_{f_1',f_2',f_3'}]{} & f_1' \circ (f_2' \circ f_3') \end{array} \tag{4.2.2}$$

We also require the following diagrams of 2-arrows to commute:

$$\begin{array}{ccc} (f_1 \circ 1) \circ f_2 & \overset{\text{ass}}{\Longrightarrow} & f_1 \circ (1 \circ f_2) \\ & {}_{r_{f_1} \bullet 1_{f_2}} \searrow \quad \swarrow {}_{1_{f_1} \bullet l_{f_2}} & \\ & f_1 \circ f_2 & \end{array} \tag{4.2.3}$$

for all pairs of composable arrows $f_1, f_2$ in $\mathcal{C}$, and

$$\begin{array}{ccc} ((f_1 \circ f_2) \circ f_3) \circ f_4 & \overset{\text{ass}}{\Longrightarrow} & (f_1 \circ f_2) \circ (f_3 \circ f_4) \\ \Downarrow \text{ass} & & \Downarrow \text{ass} \\ (f_1 \circ (f_2 \circ f_3)) \circ f_4 & & f_1 \circ (f_2 \circ (f_3 \circ f_4)) \\ & \underset{\text{ass}}{\Longrightarrow} \; f_1 \circ ((f_2 \circ f_3) \circ f_4) \; \overset{\text{ass}}{\Longrightarrow} & \end{array} \tag{4.2.4}$$

for all quadruples of composable arrows.

A bicategory is called *strictly unital* if all uniters $l_f$ and $r_f$ are identity maps, and *strict* if all uniters and associators are identities.

EXAMPLE 4.2.2. The coherence conditions (4.2.3) and (4.2.4) are trivial if all uniters and associators are identities. Thus a strict bicategory is the same as a 2-category. In particular, the 2-category $\mathcal{C}^*(2)$ introduced in Chapter 2 is a strict bicategory. Since any category becomes a 2-category by taking only identity 2-arrows, it also becomes a strict bicategory.

DEFINITION 4.2.3. The bicategory $\mathfrak{Rings}$ has rings as objects, $S, R$-bimodules as arrows $S \leftarrow R$, and bimodule maps between them as 2-arrows. The *vertical product* of arrows is the composition of bimodule maps. The *product* of arrows and the *horizontal product* of 2-arrows are $\otimes_R$. The *uniters* and the *associator* are the canonical isomorphisms

$$\begin{aligned} S \otimes_S Q &\cong Q, & s \otimes q &\mapsto s \cdot q, \\ Q \otimes_R R &\cong Q, & q \otimes r &\mapsto q \cdot r, \\ (O \otimes_T P) \otimes_S Q &\cong O \otimes_T (P \otimes_S Q), & (o \otimes p) \otimes q &\mapsto o \otimes (p \otimes q). \end{aligned}$$

It is easy to check that $\mathfrak{Rings}$ is indeed a bicategory. This example of a bicategory is already mentioned by Bénabou in [**6**, Section 2.5]. The associators and uniters in $\mathfrak{Rings}$ are already mentioned in [**19**, Equations (3.8) and (3.9)]. Notice that an $S, R$-bimodule is an arrow from $R$ to $S$. This convention causes some confusion. The other direction for the arrows would, however, also cause confusion in other places, and we prefer the convention above.

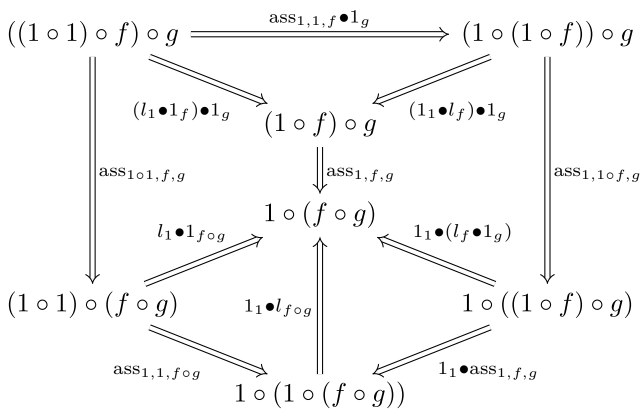
We will later make precise in which sense $\mathfrak{Rings}$ is equivalent to the 2-category of colimit-preserving functors between module categories (see Example **??**). More generally, Theorem 4.9.3 says that any bicategory is equivalent to a 2-category.

LEMMA 4.2.4 ([**14**, Theorem 7]). *Let $f_1 \colon y \to z$ and $f_2 \colon x \to y$ be composable arrows in a bicategory $\mathcal{C}$. Then the following diagrams commute:*

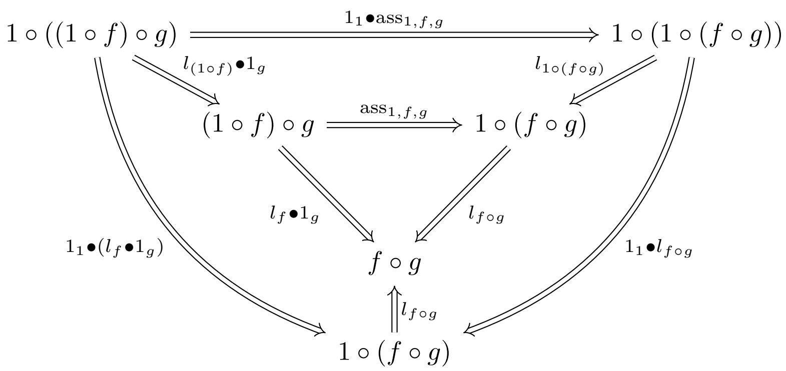
$$\begin{array}{ccc} (1_z \circ f_1) \circ f_2 & \overset{\text{ass}}{\Longrightarrow} & 1 \circ (f_1 \circ f_2) \\ & {}_{l_{f_1} \bullet 1_{f_2}} \searrow \quad \swarrow {}_{l_{f_1 \circ f_2}} & \\ & f_1 \circ f_2 & \end{array} \qquad \begin{array}{ccc} (f_1 \circ f_2) \circ 1_x & \overset{\text{ass}}{\Longrightarrow} & f_1 \circ (f_2 \circ 1) \\ & {}_{r_{f_1 \circ f_2}} \searrow \quad \swarrow {}_{1_{f_1} \bullet r_{f_2}} & \\ & f_1 \circ f_2 & \end{array} \tag{4.2.5}$$

*Thus it would make no difference to add them in Definition* 4.2.1. *The two diagrams are called the left and right triangle identity.*

PROOF. First, we consider the diagram




The outer pentagon commutes by the pentagon identity. The upper triangle and lower left triangle commute by the middle triangle identity. The two quadrilaterals commute because $a$ is natural. Therefore, the lower right triangle commutes. Thus, the outer boundary in the following diagram commutes:




The quadrilaterals of this diagram are naturality squares of $l$. Therefore, the inner triangle commutes. This is the left triangle identity. The proof that the right triangle identity commutes is similar. □

DEFINITION 4.2.5. Let $\mathcal{C}$ be a bicategory. A 2-arrow $\alpha\colon f \Rightarrow g$ in $\mathcal{C}$ is *invertible* if there is a 2-arrow $\alpha^{-1}\colon g \Rightarrow f$ with $\alpha^{-1}\cdot\alpha = 1_f$ and $\alpha\cdot\alpha^{-1} = 1_g$. Two arrows $f, g$ in $\mathcal{C}$ are *isomorphic* if there is an invertible 2-arrow $\alpha\colon f \Rightarrow g$; we write $f \cong g$. An *equivalence* between two objects $x$ and $y$ in $\mathcal{C}$ is an arrow $\alpha\colon x \to y$ for which there is an arrow $\beta\colon y \to x$ such that $\beta\circ\alpha \cong 1_x$ and $\alpha\circ\beta \cong 1_y$. Implicitly, this contains invertible 2-arrows $\beta\circ\alpha \Rightarrow 1_x$ and $\alpha\circ\beta \Rightarrow 1_y$. We call $x$ and $y$ *equivalent* and write $x \simeq y$ if there is an equivalence between them.

EXAMPLE 4.2.6. In the 2-category $\mathfrak{Cat}$ of categories, an invertible 2-arrow is the same as a natural isomorphism between two functors, and an equivalence is the same as a functor that is an equivalence of categories. In the 2-category $\mathfrak{Rings}$, an invertible 2-arrow is the same as an isomorphism of bimodules. And an equivalence is the same as a bimodule that is part of a Morita equivalence.

EXERCISE 4.2.7. *Let* $\mathcal{C}$ *be a bicategory. Let* $\mathcal{C}'$ *be the set of isomorphism classes of arrows for the isomorphism relation introduced in Definition* 4.2.5. *Show that* *there is a category with object set* $\mathcal{C}^0$ *and set of arrows* $\mathcal{C}'$*, with the product defined by* $[f] \circ [g] := [f \circ g]$ *for composable arrows* $f, g$ *in* $\mathcal{C}$*. Show that an arrow* $f$ *in* $\mathcal{C}$ *is an equivalence if and only if its image in* $\mathcal{C}'$ *is invertible.*

Invertible arrows in a bicategory are equivalences. By Exercise 4.2.7, an arrow that is isomorphic to an equivalence is an equivalence as well. In addition, the set of equivalences in a bicategory enjoys the following properties:

LEMMA 4.2.8. *Let* $\mathcal{C}$ *be a bicategory and let* $f\colon x_3 \to x_4$*,* $g\colon x_2 \to x_3$ *and* $h\colon x_1 \to x_2$ *be composable arrows in* $\mathcal{C}$*. If two of* $f$*,* $g$ *and* $f \circ g$ *are equivalences, then so is the third; this is the* 2-out-of-3 property*. If* $f \circ g$ *and* $g \circ h$ *are equivalences, then so are* $f$*,* $g$*,* $h$*, and* $f \circ g \circ h$*; this is the* 2-out-of-6 property*. The* 2*-out-of-*6 *property implies the* 2*-out-of-*3 *property.*

PROOF. First we check that the set of isomorphisms in a category has the 2-out-of-6 property. Let $(fg)^{-1}$ and $(gh)^{-1}$ be inverse to $fg$ and $gh$, respectively. Then $(fg)^{-1}fg = 1$ and $gh(gh)^{-1} = 1$ imply that $g$ is both left and right invertible. Then $g$ is invertible. Then $f = fg \cdot g^{-1}$ and $h = g^{-1} \cdot gh$ are invertible, and so is $fgh$. If a set of arrows in a category contains all identities and has the 2-out-of-6 property, then it has the 2-out-of-3 property as well. To see this, take $f = 1$, $g = 1$ or $h = 1$ in the 2-out-of-6 property and denote the remaining two arrows by $f, g$.

Exercise 4.2.7 shows that an arrow in the bicategory $\mathcal{C}$ is an equivalence if and only if its image in the truncated category $\mathcal{C}'$ is an isomorphism. Thus equivalences enjoy the 2-out-of-6 and 2-out-of-3 properties. □

In particular, equivalences of categories and Morita equivalences of rings have the 2-out-of-3 and 2-out-of-6 properties. These two properties are expected for a set of "weak equivalences" in a category (see [**23**, Definition 6.4.1]).

The following exercise generalises the observation in Exercise 4.1.19.

EXERCISE 4.2.9. *Let* $\mathcal{C}$ *be a bicategory. Let* $\alpha\colon x \to y$ *be an equivalence in* $\mathcal{C}$*. Choose* $\beta\colon y \to x$ *with* $\beta \circ \alpha \cong 1_x$ *and* $\alpha \circ \beta \cong 1_y$*. Then the invertible* 2*-arrows* $1_x \Rightarrow \beta \circ \alpha$ *and* $\alpha \circ \beta \Rightarrow 1_y$ *may be chosen so that the resulting composite* 2*-arrows*

$$\alpha \cong \alpha \circ 1_x \Rightarrow \alpha \circ (\beta \circ \alpha) \cong (\alpha \circ \beta) \circ \alpha \Rightarrow 1_y \circ \alpha \cong \alpha,$$
$$\beta \cong 1_x \circ \beta \Rightarrow (\beta \circ \alpha) \circ \beta \cong \beta \circ (\alpha \circ \beta) \Rightarrow \beta \circ 1_y \cong \beta$$

*are both unit* 2*-arrows.* (*See also* [**11**, Section 1].) *The arrows* $\alpha$ *and* $\beta$ *together with* 2*-arrows with these properties are called an* adjoint equivalence*.*

## 4.3. Morphisms, transformations, icons, modifications

We now define morphisms between bicategories, transformations and icons between these morphisms, and modifications between transformations. A morphism from a group to $\mathcal{C}^*(2)$ is the same as a weakened group action as defined in Section 2.4, and transformations and modifications between them have the same meaning as in Section 2.4. Here groups may also be replaced by crossed modules. Icons generalise cocycle-equivalences between twisted actions of groups and crossed modules. We will work out in Section 4.4 what these concepts give for the bicategory $\mathfrak{Rings}$.

### 4.3.1. Morphisms and homomorphisms.

DEFINITION 4.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. A *morphism* $\mathcal{C} \to \mathcal{D}$ consists of

- a function $F^0\colon \mathcal{C}^0 \to \mathcal{D}^0$ between the objects;
- functors $F\colon \mathcal{C}(x, y) \to \mathcal{D}(F^0(x), F^0(y))$ for all objects $x, y \in \mathcal{C}^0$;
- natural 2-arrows $\mu_{f,g}\colon F(f) \circ F(g) \Rightarrow F(f \circ g)$ for all composable arrows $f, g$ in $\mathcal{C}$; and
- 2-arrows $\lambda_x\colon 1_{F^0(x)} \Rightarrow F(1_x)$ for all objects $x \in \mathcal{C}^0$;

such that the following diagrams commute:

(4.3.1)
$$\begin{array}{ccccc} (Ff \circ Fg) \circ Fh & \xRightarrow{\mu_{f,g} \bullet 1_{Fh}} & F(f \circ g) \circ Fh & \xRightarrow{\mu_{f \circ g,h}} & F((f \circ g) \circ h) \\ \Downarrow {\scriptstyle \mathrm{ass}_{Ff,Fg,Fh}} & & & & \Downarrow {\scriptstyle F(\mathrm{ass}_{f,g,h})} \\ Ff \circ (Fg \circ Fh) & \xRightarrow{1_{Ff} \bullet \mu_{g,h}} & Ff \circ F(g \circ h) & \xRightarrow{\mu_{f,g \circ h}} & F(f \circ (g \circ h)) \end{array}$$

for three composable arrows $f, g, h$ and

(4.3.2)
$$\begin{array}{ccc} Ff \circ 1_{F^0x} & \xRightarrow{1_{Ff} \bullet \lambda_x} & Ff \circ F(1_x) \\ \Downarrow {\scriptstyle r_{Ff}} & & \Downarrow {\scriptstyle \mu_{f,1_x}} \\ Ff & \xLeftarrow{F(r_f)} & F(f \circ 1_x) \end{array} \qquad \begin{array}{ccc} 1_{F^0y} \circ Ff & \xRightarrow{\lambda_y \bullet 1_{Ff}} & F(1_y) \circ Ff \\ \Downarrow {\scriptstyle l_{Ff}} & & \Downarrow {\scriptstyle \mu_{1_y,f}} \\ Ff & \xLeftarrow{F(l_f)} & F(1_y \circ f) \end{array}$$

for an arrow $f\colon x \to y$ in $\mathcal{C}$.

A morphism is a *homomorphism* if the 2-arrows $\mu_{g,h}$ and $\lambda_x$ are invertible; it is a *strict homomorphism* if the 2-arrows $\mu_{g,h}$ and $\lambda_x$ are identities; it is *strictly unital* if the 2-arrows $\lambda_x$ are identities.

The functors $\mathcal{C}(x, y) \to \mathcal{D}(F^0x, F^0y)$ in a morphism map an arrow $f\colon x \to y$ to an arrow $F(f)\colon F^0(x) \to F^0(y)$ and a 2-arrow $\alpha\colon f \Rightarrow g$ for $f, g\colon x \rightrightarrows y$ to a 2-arrow $F(\alpha)\colon F(f) \Rightarrow F(g)$. This must be functorial for the vertical product and preserve unit 2-arrows. The naturality of the maps $\mu_{f,g}$ says that the following diagram commutes for any 2-arrows $\alpha\colon f \Rightarrow f'$, $\beta\colon g \Rightarrow g'$ in $\mathcal{C}$ with composable arrows $f, g$:

(4.3.3)
$$\begin{array}{ccc} Ff \circ Fg & \xRightarrow{\mu_{f,g}} & F(f \circ g) \\ \Downarrow {\scriptstyle F(\alpha) \bullet F(\beta)} & & \Downarrow {\scriptstyle F(\alpha \bullet \beta)} \\ Ff' \circ Fg' & \xRightarrow{\mu_{f',g'}} & F(f' \circ g') \end{array}$$

There is no naturality condition for the arrows $\lambda_x$.

If all 2-arrows in $\mathcal{D}$ are invertible, then there is no difference between morphisms and homomorphisms to $\mathcal{D}$. This happens, in particular, if $\mathcal{D} = \mathcal{C}^*(2)$. And if $\mathcal{D}$ is strict like $\mathcal{C}^*(2)$, then the associators and uniters in it are identities and may be left out in the diagrams above. If $\mathcal{C}$ is just a category, then the naturality assumption for 2-arrows above is empty.

EXAMPLE 4.3.2. A morphism from a group $G$ to $\mathcal{C}^*(2)$ is the same as a weakened group action. For a morphism $G \to \mathcal{C}^*(2)$, the diagram (4.3.1) simplifies to a commuting square as in Definition 2.3.2.(3), and the two diagrams in (4.3.2) simplify to the coherence conditions in (2.4.1); these are equivalent to Definition 2.3.2.(1) in the presence of Definition 2.3.2.(3). The naturality of the 2-arrows $\mu_{f,g}$ is empty because $G$ has only identity 2-arrows. Since all arrows in $\mathcal{C}^*(2)$ are invertible, any morphism is a homomorphism. Strictly unital homomorphisms $G \to \mathcal{C}^*(2)$ correspond to twisted actions. And strict homomorphisms are equivalent to ordinary untwisted group actions.

EXERCISE 4.3.3. *Let $(G, H, \partial, c)$ be a crossed module and let $\mathcal{C}$ be the corresponding 2-group (2-category with one object and only invertible arrows and 2-arrows). Identify strictly unital homomorphisms and strict homomorphisms from $\mathcal{C}$ to $\mathcal{C}^*(2)$ with twisted actions and actions of $\mathcal{C}$ on C*-algebras, respectively (see Definition 2.3.2 and Definition 2.7.2).*

EXERCISE 4.3.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism. Show that $F$ maps invertible 2-arrows to invertible 2-arrows. Thus $F$ preserves the isomorphism relation on arrows.*

Exercise 4.3.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a homomorphism. Show that $F$ descends to a functor between the categories associated to $\mathcal{C}$ and $\mathcal{D}$ in Exercise 4.2.7. Deduce that $F$ maps an equivalence $f\colon x \to y$ in $\mathcal{C}$ to an equivalence in $\mathcal{D}$. Thus $F^0(x)$ and $F^0(y)$ are equivalent in $\mathcal{D}$ if $x$ and $y$ are equivalent in $\mathcal{C}$.*

Exercise 4.3.5 only works for homomorphisms. If $F\colon \mathcal{C} \to \mathcal{D}$ is only a morphism, then it need not preserve equivalence of objects.

Proposition 4.3.6. *Let $\mathcal{C}$ be a bicategory and let $F\colon \mathcal{C} \to \mathfrak{Rings}$ be a homomorphism. If $x, y \in \mathcal{C}^0$ and an arrow $f \in \mathcal{C}(x, y)$ is an equivalence, then the $F^0(x), F^0(y)$-bimodule $F(f)$ is a finitely generated, projective generator both as a right $F^0(y)$-module and as a left $F^0(x)$-module. Even more, $F(f)$ is a Morita equivalence bimodule.*

Proof. Exercise 4.3.5 says that $F(f)$ is an equivalence in $\mathfrak{Rings}$. This is the same as a Morita equivalence by Exercise 4.2.6. Then $F(f)$ is a finitely generated, projective generator as a right $F^0(y)$-module by Theorem 4.1.16. For symmetry reasons, the same holds for $F(f)$ as a left $F^0(x)$-module. □

### 4.3.2. Transformations and strong transformations.

Definition 4.3.7. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be morphisms. A *transformation* $F \Rightarrow G$ consists of

- arrows $\sigma_x\colon F^0(x) \to G^0(x)$ for all $x \in \mathcal{C}^0$;
- natural 2-arrows $\sigma_f\colon G(f) \circ \sigma_x \Rightarrow \sigma_y \circ F(f)$ for all arrows $f\colon x \to y$ in $\mathcal{C}$;

such that the diagrams
(4.3.4)

$$\begin{array}{ccccccc}
(Gf \circ Gg) \circ \sigma_x & \overset{\text{ass}}{\Longrightarrow} & Gf \circ (Gg \circ \sigma_x) & \overset{1\bullet\sigma_g}{\Longrightarrow} & Gf \circ (\sigma_y \circ Fg) & \overset{\text{ass}^{-1}}{\Longrightarrow} & (Gf \circ \sigma_y) \circ Fg \\
{\scriptstyle \mu^G_{f,g}\bullet 1}\Downarrow & & & & & & \Downarrow{\scriptstyle \sigma_f\bullet 1} \\
G(f \circ g) \circ \sigma_x & \overset{\sigma_{f\circ g}}{\Longrightarrow} & \sigma_z \circ F(f \circ g) & \overset{1\bullet\mu^F_{f,g}}{\Longleftarrow} & \sigma_z \circ (Ff \circ Fg) & \overset{\text{ass}}{\Longleftarrow} & (\sigma_z \circ Ff) \circ Fg
\end{array}$$

commute for all composable arrows $f\colon y \to z$, $g\colon x \to y$ in $\mathcal{C}$, and the diagrams

$$\begin{array}{ccccc}
1_{G^0x} \circ \sigma_x & \overset{l_{\sigma_x}}{\Longrightarrow} & \sigma_x & \overset{r^{-1}_{\sigma_x}}{\Longrightarrow} & \sigma_x \circ 1_{F^0x} \\
{\scriptstyle \lambda^G_x\bullet 1_{\sigma_x}}\Downarrow & & & & \Downarrow{\scriptstyle 1_{\sigma_x}\bullet\lambda^F_x} \\
G(1_x) \circ \sigma_x & & \overset{\sigma_{1_x}}{\Longrightarrow} & & \sigma_x \circ F(1_x)
\end{array} \tag{4.3.5}$$

commute for all objects $x \in \mathcal{C}^0$.

A transformation is *strong* if all the 2-arrows $\sigma_f$ are invertible, and *strict* if all the 2-arrows $\sigma_f$ are identities.

The naturality of the 2-arrows $\sigma_f$ says that if $x, y \in \mathcal{C}^0$, $f, g \in \mathcal{C}(x, y)$ and $\alpha\colon f \Rightarrow g$ is a 2-arrow in $\mathcal{C}$, then the following diagram of 2-arrows commutes:

$$\begin{array}{ccc}
G(f) \circ \sigma_x & \overset{\sigma_f}{\Longrightarrow} & \sigma_y \circ F(f) \\
{\scriptstyle G(\alpha)\bullet 1_{\sigma_x}}\Downarrow & & \Downarrow{\scriptstyle 1_{\sigma_y}\bullet F(\alpha)} \\
G(g) \circ \sigma_x & \overset{\sigma_g}{\Longrightarrow} & \sigma_y \circ F(g)
\end{array} \tag{4.3.6}$$

This is trivial if $\alpha$ is an identity 2-arrow. Therefore, the naturality of the 2-arrows $\sigma_f$ above is empty if $\mathcal{C}$ is a category, viewed as a bicategory.

If $\mathcal{D}$ is a 2-category like $\mathcal{C}^*(2)$, then the two coherence diagrams for a transformation simplify because the associators and uniters in $\mathcal{D}$ may be left out, being unit 2-arrows. This is why our previous definitions of a transformation between twisted actions of groups and crossed modules in Definition 2.4.3 and Definition 2.7.3 are special cases of Definition 4.3.7:

EXERCISE 4.3.8. *Let* $\mathcal{C}$ *be the* 2*-category associated to a group or, more generally, to a crossed module. Identify two strictly unital homomorphisms* $\alpha, \beta \colon \mathcal{C} \rightrightarrows \mathcal{C}^*(2)$ *with twisted group actions as in Example* 4.3.2 *and Exercise* 4.3.3*. Show that a transformation* $\alpha \Rightarrow \beta$ *is the same as a transformation between the corresponding twisted actions* (*see Definition* 2.4.3 *and Definition* 2.7.3).

If all 2-arrows in $\mathcal{D}$ are invertible, then any transformation is strong. This is why we have not yet seen the difference between transformations and strong transformations in $\mathcal{C}^*(2)$. The following proposition gives a less obvious sufficient condition for all transformations to be strong:

PROPOSITION 4.3.9. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be bicategories. Assume that any arrow in* $\mathcal{C}$ *is an equivalence. Let* $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ *be homomorphisms. Then any transformation* $F \Rightarrow G$ *is strong.*

PROOF. A transformation $\sigma \colon F \Rightarrow G$ consists of arrows $\sigma_x \colon F^0(x) \to G^0(x)$ for $x \in \mathcal{C}^0$ and natural 2-arrows $\sigma_f \colon G(f) \circ \sigma_x \Rightarrow \sigma_y \circ F(f)$ for all arrows $f \colon x \to y$ in $\mathcal{C}$, subject to the coherence conditions in Definition 4.3.7. The claim is that $\sigma_f$ is invertible. We are going to prove that $\sigma_f$ is left invertible. A similar argument shows that it is right invertible, and then it is invertible.

Let $x \in \mathcal{C}^0$. Since $F$ and $G$ are homomorphisms, all 2-arrows in the coherence diagram (4.3.5) except $\sigma_{1_x}$ are invertible. Thus $\sigma_{1_x}$ is invertible. Let $f \colon x \to y$ in $\mathcal{C}$. By assumption, $f$ is an equivalence. So there is an arrow $g \colon y \to x$ such that $g \circ f \cong 1_x$. This gives an invertible 2-arrow $\alpha \colon g \circ f \Rightarrow 1_x$. Both $F$ and $G$ map it to an invertible 2-arrow in $\mathcal{D}$ by Exercise 4.3.4. Thus the vertical arrows in the naturality diagram (4.3.6) for $\alpha$ are invertible. Then $\sigma_{g \circ f}$ is invertible because $\sigma_{1_x}$ is invertible. Since $F$ and $G$ are homomorphisms, the 2-arrows $\mu^G_{g,f}$ and $\mu^F_{g,f}$ are invertible, and so are the various associators in (4.3.4). Since $\sigma_{g \circ f}$ is invertible, it follows that $1_{G(g)} \bullet \sigma_f$ is left invertible. Then so is

$$1_{G(f) \circ G(g)} \bullet \sigma_f = (1_{G(f)} \bullet 1_{G(g)}) \bullet \sigma_f = \mathrm{ass}^{-1} \circ \big(1_{G(f)} \bullet (1_{G(g)} \bullet \sigma_f)\big) \circ \mathrm{ass};$$

the last equality uses the naturality of associators. Then $1_{G^0(y)} \bullet \sigma_f$ is left invertible because of the invertible 2-arrows

$$G(f) \circ G(g) \xRightarrow{\mu^G_{f,g}} G(f \circ g) \cong G(1_y) \xRightarrow{(\lambda^G_y)^{-1}} 1_{G^0(y)}.$$

And then $\sigma_f$ is left invertible because of the naturality of left uniters in $\mathcal{D}$ in (4.2.1). □

### 4.3.3. Modifications.

DEFINITION 4.3.10. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories, let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{C} \to \mathcal{D}$ be morphisms, and let $\sigma \colon F \Rightarrow G$ and $\sigma' \colon F \Rightarrow G$ be transformations. A *modification* $\Gamma \colon \sigma \Rrightarrow \sigma'$ is a collection of 2-arrows $\Gamma_x \colon \sigma_x \Rightarrow \sigma'_x$ for all $x \in \mathcal{C}^0$ making the following diagrams commute for all arrows $f \colon x \to y$ in $\mathcal{C}$:

$$\begin{array}{ccc} Gf \circ \sigma_x & \xRightarrow{1 \bullet \Gamma_x} & Gf \circ \sigma'_x \\ \Downarrow{\scriptstyle \sigma_f} & & \Downarrow{\scriptstyle \sigma'_f} \\ \sigma_y \circ Ff & \xRightarrow{\Gamma_y \bullet 1} & \sigma'_y \circ Ff \end{array}$$

EXERCISE 4.3.11. *Continuing Exercise* 4.3.8*, identify modifications between transformations between strictly unital homomorphisms from a group or crossed module to* $\mathcal{C}^*(2)$ *with the modifications defined for twisted actions* (*see Definition* 2.4.4 *and Definition* 2.7.6).

**4.3.4. Icons.** Icons are introduced by Lack [**15**], and I thank him for pointing out to me why they are useful. We may arrive at them by looking again at Table 1, which summarises classical concepts for group actions and their weakened forms in the 2-category $\mathcal{C}^*(2)$. We have seen that (strictly unital) morphisms or homomorphisms to $\mathcal{C}^*(2)$ generalise twisted group actions. And the concepts of transformations and modifications in bicategory theory also generalise the concepts with the same name for twisted group actions. There are, however, a few more lines in this table. Now we are going to generalise the concept of a cocycle-equivalence. We could consider strong transformations $\sigma\colon F \Rightarrow G$ for which the all the arrows $\sigma_x\colon F^0(x) \to G^0(x)$ are unit arrows; in particular, this says that $F^0(x) = G^0(x)$ for all $x \in \mathcal{C}^0$. Then, however, we may simplify the domain and codomain of $\sigma_f$ because there are isomorphisms of arrows $G(f) \circ 1_{G^0(x)} \cong G(f)$ and $1_{F^0(x)} \circ F(f) \cong F(f)$. When we do this, we replace $\sigma_f$ by a 2-arrow $\sigma'_f\colon G(f) \Rightarrow F(f)$. Then the coherence conditions for a transformation simplify a lot. It is more natural to reverse the direction, however, and consider a family of 2-arrows $F(f) \Rightarrow G(f)$. This is how one may arrive at the following definition:

Definition 4.3.12 ([**15**]). Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be morphisms such that $F^0(x) = G^0(x)$ for all $x \in \mathcal{C}^0$. An *icon* $\alpha\colon F \Rightarrow G$ is a family of natural 2-arrows $\alpha_f\colon F(f) \Rightarrow G(f)$ such that the following diagrams of 2-arrows commute for composable arrows $f, g$ in $\mathcal{C}$ and $x \in \mathcal{C}^0$:

$$\begin{array}{ccc} F(f)\circ F(g) & \xRightarrow{\mu^F_{f,g}} & F(f\circ g) \\ \Downarrow{\scriptstyle \alpha_f\bullet\alpha_g} & & \Downarrow{\scriptstyle \alpha_{f\circ g}} \\ G(f)\circ G(g) & \xRightarrow{\mu^G_{f,g}} & G(f\circ g) \end{array} \qquad \begin{array}{ccc} 1_{F^0(x)} & \xRightarrow{\lambda^F(x)} & F(1_x) \\ \Vert & & \Downarrow{\scriptstyle \alpha_{1_x}} \\ 1_{G^0(x)} & \xRightarrow{\lambda^G(x)} & G(1_x) \end{array}$$

Proposition 4.3.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be morphisms such that $F^0(x) = G^0(x)$ for all $x \in \mathcal{C}^0$. There is a bijection between icons $\alpha\colon F \Rightarrow G$ and transformations $\sigma = (\sigma_x, \sigma_f)\colon G \Rightarrow F$ with the extra property that $\sigma_x = 1_{F^0(x)}$ for all $x \in \mathcal{C}^0$.*

Proof. Let $\sigma_x = 1_{F^0(x)}$ for all $x \in \mathcal{C}^0$. For $x, y \in \mathcal{C}^0$ and $f \in \mathcal{C}(x, y)$, a 2-arrow $\sigma_f\colon F(f) \circ \sigma_x \Rightarrow \sigma_y \circ G(f)$ gives a 2-arrow $\alpha_f\colon F(f) \Rightarrow G(f)$, namely, the vertical product

$$F(f) \xRightarrow[\cong]{r^{-1}_{F(f)}} F(f) \circ 1_{F^0(x)} = F(f) \circ \sigma_x \xRightarrow{\sigma_f} \sigma_y \circ G(f) = 1_{G^0(y)} \circ G(f) \xRightarrow[\cong]{l_{F(f)}} G(f).$$

Since the uniters in this product are invertible, $\alpha_f$ gives back $\sigma_f$ by a similar formula. We claim that the 2-arrows $\alpha_f$ for all arrows $f$ in $\mathcal{C}$ form an icon if and only if the arrows $1_{F^0(x)}$ for $x \in \mathcal{C}^0$ and the 2-arrows $\sigma_f$ for all arrows $f$ in $\mathcal{C}$ form a transformation. Indeed, the two diagrams in Definition 4.3.12 commute if and only if the diagrams in (4.3.4) and (4.3.5) commute. To see this for (4.3.4), copy the diagram and remove each $\sigma_x$, $\sigma_y$ or $\sigma_z$ in it; the associators become identities because there are now only two arrows to compose. Since we only removed unit arrows, horizontal products with uniters give invertible 2-arrows that link the new and the old diagram. The squares that are formed in this way commute because of the diagrams in (4.2.3) and (4.2.5). As a consequence, the old diagram (4.3.4) commutes if and only if the new one associated to it does. And leaving out equalities, the latter diagram becomes the first diagram in Definition 4.3.12. The diagram in (4.3.5) and the other diagram in Definition 4.3.12 are linked in a similar way through commuting squares with invertible 2-arrows. Thus one commutes if and only if the other one does. $\square$

**4.3.5. Notation and some history.** We may arrive at bicategories, morphisms, transformations and modifications following the scheme introduced in Section 2.4, by weakening the concepts of a category, a functor, a natural transformation, and an equality of natural transformations in usual category theory. For instance, when we weaken the concept of a functor, then the 2-arrows $\mu_{f,g}$ and $\lambda_x$ replace the usual assumptions $F(f) \circ F(g) = F(f \circ g)$ and $F(1_x) = 1_{Fx}$ for a functor. And the coherence conditions come from the two ways of proving $Ff \circ Fg \circ Fh = F(f \circ g \circ h)$ and $F(f \circ 1_x) = Ff$ and $F(1_y \circ f) = Ff$ for an ordinary functor. The weakening scheme works best, however, when all 2-arrows are invertible. In a bicategory with non-invertible 2-arrows, there are three ways to weaken an equality: replace an equality of arrows by an invertible 2-arrow or an arbitrary 2-arrow in one or the other direction; and of course, we may also keep the equality. In our naming convention above (which follows Leinster [**18**]), we use the adjective "strict" in case equality is kept, and "strong" if it is replaced by an invertible 2-arrow. Without extra adjective, we have weakened equality to an arbitrary 2-arrow in one particular direction. Other authors use other names instead. One consistent way to name the various concepts is to use the prefix "2-" if equality is kept, the prefix "pseudo-" if equality is replaced by an invertible 2-arrow, and the adjective "lax" if equality is weakened to an arbitrary 2-arrow. Authors following this notation would speak of pseudo-functors and lax functors instead of homomorphisms and morphisms between bicategories. The adjective "oplax" may then be used if equality of arrows is weakened to a 2-arrow that is pointing in the non-standard direction (see [**16**, Section 1.2]). The name "pseudo-functor" goes back to Grothendieck [**10**] (see [**6**, Section 5.6] for the comparison).

Bicategories and morphisms have been defined first by Bénabou. He chose the directions of the 2-arrows in the definition of a morphism to account for the following two important examples.

Example 4.3.14. Let $\mathcal{C}$ be a bicategory with only one object, which we denote by $\star$. The bicategory structure on $\mathcal{C}$ is equivalent to a *monoidal category* structure on the category $\mathcal{C}(\star,\star)$ of endomorphisms of the unique object. Here we write the product of arrows in $\mathcal{C}$ as a tensor product. Two examples of monoidal categories are the category of vector spaces with the usual tensor product and the category of sets with the Cartesian product. So these are also bicategories with a single object. The theory of monoidal categories was developed before bicategory theory, including the concepts of (lax) monoidal functors and monoidal natural transformations.

Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, viewed as bicategories with only one object $\star$. Then a morphism $\mathcal{C} \to \mathcal{D}$ is the same as a *lax monoidal functor* $\mathcal{C} \to \mathcal{D}$. An obvious example of a lax monoidal functor that is not strong is the forgetful functor from vector spaces with the monoidal functor $\otimes$ to sets with the monoidal functor $\times$.

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be monoidal functors, viewed as morphisms of bicategories. A *monoidal natural transformation* $F \Rightarrow G$ is the same as an icon $F \Rightarrow G$. Equivalently, it is a transformation $G \Rightarrow F$ with the extra property that the arrow $\sigma_\star\colon \star \to \star$ in Definition 4.3.16 is the unit arrow on $\star$. Here the direction is reversed.

Example 4.3.15. Let $\mathbf{1}$ denote the bicategory with one object, one arrow and one 2-arrow. A *monad* in a bicategory $\mathcal{C}$ is defined as a morphism $\mathbf{1} \to \mathcal{C}$ (see [**6**, Section 5.4]); this reproduces the usual concept of a monad (see [**23**]) in the bicategory of categories $\mathfrak{Cat}$. Any morphism from a bicategory $\mathcal{C}_1$ to a bicategory $\mathcal{C}_2$ maps monads in $\mathcal{C}_1$ to monads in $\mathcal{C}_2$ (this follows from Proposition 4.7.10).

The direction of the 2-arrows in a transformation is more debatable; we briefly touch upon this issue in Section 4.3.6.

**4.3.6. Variance.** Any category $\mathcal{C}$ has an *opposite category* $\mathcal{C}^{\mathrm{op}}$ in which the direction of the arrows and the order of the product is reversed. A functor $\mathcal{C} \to \mathcal{D}$ is also a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$, and contravariant functors are defined as functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ or, equivalently, $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Similarly, a bicategory $\mathcal{C}$ has two kinds of opposites:

- $\mathcal{C}^{\mathrm{op}}$ is the bicategory where the direction of the arrows and the order of products and horizontal products is reversed;
- $\mathcal{C}^{\mathrm{co}}$ is the bicategory where the direction of the 2-arrows and the order of vertical products is reversed.

These may be combined to form the bicategory $\mathcal{C}^{\mathrm{co,op}}$ where the directions of arrows and 2-arrows and the orders of all products are reversed.

A morphism or homomorphism of bicategories $\mathcal{C} \to \mathcal{D}$ is also a morphism or homomorphism $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. And a homomorphism $\mathcal{C} \to \mathcal{D}$ also becomes a homomorphism $\mathcal{C}^{\mathrm{co}} \to \mathcal{D}^{\mathrm{co}}$. Since this requires taking the inverses of the 2-arrows $\mu_{f,g}$ and $\lambda_x$ in Definition 4.3.1, it fails for morphisms that are not homomorphisms. As a result, the duality operations above applied to $\mathcal{C}$ and $\mathcal{D}$ give 8 variants of the concept of a morphism and 4 variants of the concept of a homomorphism that differ in the order of products and the direction of arrows and 2-arrows.

Each of these 8 kinds of morphisms have their own transformations. In fact, each has two types of transformations that differ in the direction of the 2-arrows $\sigma_f$. Of course, there is again only one kind of strong transformation. The two types of lax transformations between lax morphisms are genuinely different, that is, we cannot turn one into the other by applying duality operations to the bicategories. Therefore, we need a name for the other kind of transformation:

Definition 4.3.16. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be morphisms. A *cotransformation* $F \Rightarrow G$ consists of arrows $\sigma_x\colon F^0(x) \to G^0(x)$ for all $x \in \mathcal{C}^0$ and natural 2-arrows $\sigma_f\colon \sigma_y \circ F(f) \Rightarrow G(f) \circ \sigma_x$ for all arrows $f\colon x \to y$ in $\mathcal{C}$, such that the analogues of the diagrams (4.3.4) and (4.3.5) commute.

If $\sigma_x$ and $\sigma_f$ form a strong transformation, then $\sigma_x$ and $\sigma_f^{-1}$ form a strong cotransformation, and vice versa. So the strong forms of transformations and cotransformations are essentially equivalent. For monoidal categories viewed as bicategories, cotransformations specialise to monoidal natural transformations in the same direction (see Example 4.3.14). Icons $F \Rightarrow G$ are special cotransformations $F \Rightarrow G$ by Proposition 4.3.13, whereas transformations specialise to icons $G \Rightarrow F$.

Finally, we may reverse the 2-arrow in a modification. This does not give a truly new concept, however, because it just turns a modification $\Gamma\colon \sigma \Rrightarrow \tau$ into a modification $\Gamma\colon \tau \Rrightarrow \sigma$.

**4.3.7. From ring homomorphisms to bimodules.** As a simple example of a homomorphism between bicategories, we relate the category of rings and ring homomorphisms to the bicategory $\mathfrak{Rings}$. We are going to define a strictly unital homomorphism from the opposite category of rings and ring homomorphisms to the bicategory $\mathfrak{Rings}$.

Let $R$ and $S$ be rings and let $f\colon S \to R$ be a homomorphism. It generates a functor $f^*$ from $R$- to $S$-modules as in Example 4.1.11. And this functor is naturally isomorphic to the functor that tensors with the $S,R$-bimodule $R_f$ that is $R$ as a right $R$-module with the left $S$-module structure $s \cdot r := f(s) \cdot r$. The natural isomorphism between $R_f \otimes_R \sqcup$ and $f^*$ is the family of maps $R_f \otimes_R M \cong M$, $(r \otimes m) \mapsto r \cdot m$.

Lemma 4.3.17. *The identity map on objects and the map $f \mapsto R_f$ are part of a strictly unital homomorphism from the opposite category of rings to $\mathfrak{Rings}$.*

PROOF. If $R$ is a ring, then $R_{\mathrm{id}_R}$ is $R$ with the usual bimodule structure; this is also the unit arrow on $R$ in $\mathfrak{Rings}$. Let $f\colon R \to S$ and $g\colon S \to T$ be ring homomorphisms. Then there is a canonical bimodule homomorphism

$$S_f \otimes_S T_g \cong T_{f\circ g}, \qquad s \otimes t \mapsto g(s)t.$$

It is an exercise to check that this satisfies the coherence conditions required for a (strictly unital) homomorphism in Definition 4.3.1. □

EXERCISE 4.3.18. *Let $f_1, f_2\colon S \rightrightarrows R$ be ring homomorphisms. Find a bijection between bimodule homomorphisms $R_{f_1} \to R_{f_2}$ and elements $r \in R$ that satisfy $r \cdot f_1(s) = f_2(s) \cdot r$ for all $s \in S$ ("intertwiners"). The vertical product gives the multiplication in $R$ for these intertwiners. Describe the horizontal product of intertwiners.*

The exercise shows that the 2-arrows in $\mathfrak{Rings}$ specialise to the intertwiners between ring homomorphisms when we restrict to arrows of the form $R_f$. The resulting bicategory of rings with homomorphisms as arrows and intertwiners as 2-arrows is similar to the variant of $\mathcal{C}^*(2)$ mentioned in Remark 2.2.6. The homomorphism in Lemma 4.3.17 extends to a homomorphism on the larger 2-category or rings, ring homomorphisms, and intertwiners.

## 4.4. Weakened dynamical systems on rings

We are going to describe morphisms to the bimodule bicategory $\mathfrak{Rings}$ and their transformations and modifications. Given our previous experience with $\mathcal{C}^*(2)$, morphisms from a group to $\mathfrak{Rings}$ should be something like group actions on a ring. As we shall see, these are close to graded rings. More generally, we describe morphisms from a category to $\mathfrak{Rings}$ and from a 2-category to $\mathfrak{Rings}$. We do not consider morphisms from arbitrary bicategories to $\mathfrak{Rings}$, however; these are more complicated, and we may, anyway, replace a bicategory by an equivalent 2-category.

In the computations below, a bimodule map $X \otimes_R Y \to Z$ is replaced by a map $X \times Y \to Z$ with some extra properties. This hides the associators in $\mathfrak{Rings}$ because for a function of three variables $X \times Y \times Z \to W$, we just write $f(x, y, z)$ and not $f((x, y), z)$ or $f(x, (y, z))$ – although products of sets are not strictly associative. This is how we get rid of the associators in $\mathfrak{Rings}$.

The computations in this section are mere exercises. It seems, however, that this is the first text where this is published. In particular, we are not aware of previous work where graded rings are treated as generalised dynamical systems.

**4.4.1. Graded rings and morphisms.** We need the following generalised concept of a grading:

DEFINITION 4.4.1. Let $\mathcal{C}$ be a category. A *$\mathcal{C}$-graded ring* is a possibly nonunital ring $S$ with a direct sum decomposition $S = \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ as an Abelian group, such that $S_\gamma \cdot S_\eta \subseteq S_{\gamma\cdot\eta}$ if $\gamma, \eta \in \mathcal{C}$ are composable, $S_\gamma \cdot S_\eta = 0$ if $\gamma, \eta \in \mathcal{C}$ are not composable, and there are elements $1_x \in S_x := S_{1_x}$ for all $x \in \mathcal{C}^0$ with $1_y \cdot a = a = a \cdot 1_x$ for all $x, y \in \mathcal{C}^0$, $\gamma \in \mathcal{C}(x, y)$, $a \in S_\gamma$.

EXAMPLE 4.4.2. View the set of objects $\mathcal{C}^0$ of $\mathcal{C}$ as a category with only identity arrows. A $\mathcal{C}^0$-graded ring $R$ is equivalent to a family of unital rings $R_x$ for all $x \in \mathcal{C}^0$. The direct sum $R := \bigoplus_{x\in\mathcal{C}^0} R_x$ with the pointwise multiplication is only unital when $\mathcal{C}^0$ is finite. This is why we allow $\mathcal{C}$-graded rings to be nonunital.

A $\mathcal{C}$-graded ring $S$ restricts to a $\mathcal{C}^0$-graded ring by $S|_{\mathcal{C}^0} := \bigoplus_{x\in\mathcal{C}^0} S_x$.

PROPOSITION 4.4.3. *Let $\mathcal{C}$ be a category and let $\mathcal{C}^0$ be its set of objects. View $\mathcal{C}$ as a 2-category. A morphism from $\mathcal{C}$ to the bicategory $\mathfrak{Rings}$ is "equivalent" to a $\mathcal{C}^0$-graded ring $R = \bigoplus_{x\in\mathcal{C}^0} R_x$ and a $\mathcal{C}$-graded ring $S = \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ with a $\mathcal{C}^0$-graded, nondegenerate ring homomorphism $\lambda\colon R \to S|_{\mathcal{C}^0}$, that is, $\lambda|_{R_x}\colon R_x \to S_x$ is a unital ring homomorphism for each $x \in \mathcal{C}^0$.*

*The morphism is strictly unital if and only if $\lambda$ is the identity map; then $R$ and $\lambda$ may be left out, making a strictly unital morphism $\mathcal{C} \to \mathfrak{Rings}$ equivalent to a $\mathcal{C}$-graded ring $S$. The data above corresponds to a homomorphism if and only if $\lambda\colon R \to S|_{\mathcal{C}^0}$ is an isomorphism and the multiplication maps induce isomorphisms $S_\gamma \otimes_{S_{s(\gamma)}} S_\eta \xrightarrow{\simeq} S_{\gamma\eta}$ for composable $\gamma, \eta \in \mathcal{C}$.*

The word "equivalent" in the statement above will be made precise later, see Remark 4.5.8. For the time being, we understand the proposition as saying that we can go back and forth between morphisms and the triples $(R, S, \lambda)$ and that these constructions preserve all "important" information.

PROOF. A morphism $\mathcal{C} \to \mathfrak{Rings}$ consists of (unital) rings $R_x$ for $x \in \mathcal{C}^0$, $R_{r(\gamma)}, R_{s(\gamma)}$-bimodules $S_\gamma$ for $\gamma \in \mathcal{C}$, $R_x, R_x$-bimodule maps $\lambda_x\colon R_x \to S_x := S_{1_x}$ for $x \in \mathcal{C}^0$, and $R_{r(\gamma)}, R_{s(\eta)}$-bimodule maps $\mu_{\gamma,\eta}\colon S_\gamma \otimes_{R_{s(\gamma)}} S_\eta \to S_{\gamma\eta}$ for composable $(\gamma, \eta)$ in $\mathcal{C}$, such that the coherence diagrams in (4.3.1) and (4.3.2) commute.

Define a product on the Abelian group $S := \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ by

$$x \cdot y := \begin{cases} \mu_{\gamma,\eta}(x \otimes y) & \text{if } x \in S_\gamma,\ y \in S_\eta,\ (\gamma, \eta) \text{ composable in } \mathcal{C}, \\ 0 & \text{if } x \in S_\gamma,\ y \in S_\eta,\ (\gamma, \eta) \text{ not composable in } \mathcal{C}. \end{cases}$$

This multiplication is associative if and only if the diagram (4.3.1) commutes. And the two diagrams in (4.3.2) commute if and only if the $R_{r(\gamma)}, R_{s(\gamma)}$-bimodule structure on $S_\gamma$ is of the form $a_1 \cdot b_2 \cdot a_3 = \lambda_{r(\gamma)}(a_1) \cdot b_2 \cdot \lambda_{s(\gamma)}(a_3)$ for all $a_1 \in R_{r(\gamma)}$, $b_2 \in S_\gamma$, $a_3 \in R_{s(\gamma)}$; here the product $\lambda_{r(\gamma)}(a_1) \cdot b_2 \cdot \lambda_{s(\gamma)}(a_3)$ is taken in the ring $S$. Then $\lambda_{r(\gamma)}(1_{r(\gamma)})$ acts like a left unit on $S_\gamma$ and $\lambda_{s(\gamma)}(1_{s(\gamma)})$ acts like a right unit on $S_\gamma$. So $S$ with the decomposition above is a $\mathcal{C}$-graded ring. And $\lambda_x$ for $x \in \mathcal{C}^0$ is a unital $R_x$-bimodule map. This is equivalent to being a unital ring homomorphism. These homomorphisms for $x \in \mathcal{C}^0$ combine to a nondegenerate $\mathcal{C}^0$-graded ring homomorphism $\lambda\colon R \to S|_{\mathcal{C}^0}$ as in the statement of the proposition. And $\lambda$ pins down the $R_{r(\gamma)}, R_{s(\gamma)}$-bimodule structures on $S_\gamma$ for $\gamma \in \mathcal{C}$. This proves the description of morphisms in the first paragraph of the proposition.

By definition, a morphism is strictly unital if and only if $\lambda_x$ is an identity map for each $x \in \mathcal{C}^0$. In this case, leaving out the rings $R_x$ and the identity maps $\lambda_x$ loses no information. This gives the description of strictly unital morphisms.

And for a homomorphism, we ask that $\lambda_x\colon R_x \to S_x$ and the maps $\mu_{\gamma,\eta}$ for composable $(\gamma, \eta)$ in $\mathcal{C}$ be isomorphisms. If $\lambda_{s(\gamma)}$ is an isomorphism, then $S_\gamma \otimes_{S_{s(\gamma)}} S_\eta \cong S_\gamma \otimes_{R_{s(\gamma)}} S_\eta$. And the map $\mu_{\gamma,\eta}$ induces the same map $S_\gamma \otimes_{S_{s(\gamma)}} S_\eta \to S_{\gamma\eta}$ as the multiplication map in $S$. This proves the claims about homomorphisms. □

Proposition 4.4.3 suggests that strictly unital morphisms $\mathcal{C} \to \mathfrak{Rings}$ are more attractive than general morphisms – they are equivalent simply to $\mathcal{C}$-graded rings.

A "graded ring" commonly means a ring graded by the monoid $(\mathbb{N}, +)$, that is, $R = \bigoplus_{n=0}^{\infty} R_n$ with $R_j \cdot R_k \subseteq R_{jk}$. View the monoid $\mathbb{N}$ as a 2-category with one object and only unit 2-arrows (see Example 4.2.2). Proposition 4.4.3 says that strictly unital morphisms $\mathbb{N} \to \mathfrak{Rings}$ are equivalent to graded rings.

A strictly unital *homo*morphism $\mathbb{N} \to \mathfrak{Rings}$ has the extra property that multiplication induces isomorphisms $R_n \otimes_{R_0} R_m \xrightarrow{\simeq} R_{n+m}$ for all $n, m \in \mathbb{N}$. Then the $n$-fold multiplication map induces an isomorphism $R_1^{\otimes_{R_0} n} \xrightarrow{\simeq} R_n$ for all $n \in \mathbb{N}$. After identifying $R_n$ with this $n$-fold tensor product, the multiplication maps become just concatenation of tensors – up to associators of bimodule tensor products, which we ignore because they are so canonical. So a strictly unital homomorphism $\mathbb{N} \to \mathfrak{Rings}$ is determined by the ring $R_0$ and the $R_0$-bimodule $R_1$. We should have expected this because $\mathbb{N}$ is the free monoid on one generator. The $\mathbb{N}$-graded ring corresponding to a strictly unital homomorphism $\mathbb{N} \to \mathfrak{Rings}$ is called the tensor algebra of the bimodule $R_1$ over $R_0$.

Graded rings occur in many branches of algebra. For instance, in algebraic geometry, projective varieties are described through certain commutative graded rings. In algebraic geometry, the degree-0 component $R_0 \subseteq R$ is just the ground field. It seems far-fetched to interpret a graded ring as a generalised action of $\mathbb{N}$ on $R_0$. A class of strictly unital homomorphisms $\mathbb{N} \to \mathfrak{Rings}$ where this interpretation is fruitful is studied in Section 4.6.3.

Now we replace $\mathcal{C}$ by a 2-category. Assuming the multiplication in $\mathcal{C}$ to be strictly associative means that forgetting the 2-arrows in $\mathcal{C}$ gives a category $\mathcal{C}^{\leq 1}$. A morphism $\mathcal{C} \to \mathfrak{Rings}$ restricts to a morphism $\mathcal{C}^{\leq 1} \to \mathfrak{Rings}$. It remains to examine what the 2-arrows in $\mathcal{C}$ do:

Proposition 4.4.4. *Let $\mathcal{C}$ be a 2-category. Let $\mathcal{C}^0$ be its set of objects and $\mathcal{C}^{\leq 1}$ the category of its objects and arrows. Describe a morphism from $\mathcal{C}^{\leq 1}$ to $\mathfrak{Rings}$ through a $\mathcal{C}^0$-graded ring $R = \bigoplus_{x \in \mathcal{C}^0} R_x$ and a $\mathcal{C}^{\leq 1}$-graded ring $S = \bigoplus_{\gamma \in \mathcal{C}^{\leq 1}} S_\gamma$ with a $\mathcal{C}^0$-graded, nondegenerate ring homomorphism $\lambda\colon R \to S|_{\mathcal{C}^0}$ as in Proposition 4.4.3. A morphism $\mathcal{C} \to \mathfrak{Rings}$ is equivalent to this data together with a family of group homomorphisms $\sigma(b)\colon S_\gamma \to S_\eta$ for all 2-arrows $b\colon \gamma \Rightarrow \eta$ in $\mathcal{C}$ that is compatible with units and vertical and horizontal products; that is, $\sigma(1_\gamma) = \mathrm{id}_{S_\gamma}$ for each arrow $\gamma$; if $b\colon \gamma_1 \Rightarrow \gamma_2$ and $c\colon \gamma_2 \Rightarrow \gamma_3$ are vertically composable 2-arrows in $\mathcal{C}$, then*

$$\sigma(c \cdot b)(s) = \sigma(c)\bigl(\sigma(b)(s)\bigr) \qquad \text{for all } s \in S_{\gamma_1}.$$

*And if $\eta\colon x \to y$, $\gamma_1, \gamma_2\colon y \rightrightarrows z$, $\xi\colon z \to w$ are arrows in $\mathcal{C}$ and $b\colon \gamma_1 \Rightarrow \gamma_2$ is a 2-arrow in $\mathcal{C}$, then*

$$\sigma(1_\xi \bullet b)(s_1 \cdot s_2) = s_1 \cdot \sigma(b)(s_2), \qquad \sigma(b \bullet 1_\eta)(s_2 \cdot s_3) = \sigma(b)(s_2) \cdot s_3$$

*for all $s_1 \in S_\xi$, $s_2 \in S_{\gamma_1}$, $s_3 \in S_\eta$. If the maps $\sigma(b)$ satisfy this, then they are automatically $S_{r(\gamma)}, S_{s(\gamma)}$-bimodule homomorphisms.*

Proof. The data of a morphism $\mathcal{C} \to \mathfrak{Rings}$ contains the data of a morphism $\mathcal{C}^{\leq 1} \to \mathfrak{Rings}$. In addition, the morphism on $\mathcal{C}$ maps each 2-arrow $b\colon \gamma \Rightarrow \eta$ in $\mathcal{C}$ to a 2-arrow $\sigma(b)\colon S_\gamma \Rightarrow S_\eta$ in $\mathfrak{Rings}$, in such a way that unit 2-arrows and vertical products are preserved – this is the functoriality of $F$ in Definition 4.3.7 – and the 2-arrows $\mu_{\gamma,\eta}$ in the definition of a morphism are natural. The functoriality of $F$ is equivalent to the formulas for $\sigma(1_\gamma)$ and $\sigma(c \cdot b)$ in the statement of the proposition. By definition, each $\sigma(b)$ is an $R_{r(\gamma)}, R_{s(\gamma)}$-bimodule map. Let $\gamma_1, \gamma_2\colon y \rightrightarrows z$ be parallel arrows in $\mathcal{C}$. The condition in the proposition for $\xi = 1_z$ and $\eta = 1_y$ implies that $\sigma(b)$ for $b\colon \gamma_1 \Rightarrow \gamma_2$ is an $S_z, S_y$-bimodule homomorphism. This is stronger than being an $R_z, R_y$-bimodule homomorphism. So it suffices to assume each $\sigma(b)$ to be a group homomorphism. We still have to show that the naturality of the multiplication 2-arrows $\mu_{\gamma,\eta}$ is equivalent to the conditions about $1_\xi \bullet b$ and $b \bullet 1_\eta$ in the proposition.

To begin with, the naturality of $\mu_{\gamma,\eta}$ says the following. Let $\eta_1, \eta_2\colon x \rightrightarrows y$ and $\gamma_1, \gamma_2\colon y \rightrightarrows z$ be parallel pairs of arrows that are composable and let $b\colon \eta_1 \Rightarrow \eta_2$ and $c\colon \gamma_1 \Rightarrow \gamma_2$ be 2-arrows. Then the horizontal product $c \bullet b$ is defined. And the naturality of $\mu_{\gamma,\eta}$ says that the following diagram of bimodule homomorphisms commutes:

$$\begin{array}{ccc} S_{\gamma_1} \otimes_{R_y} S_{\eta_1} & \xrightarrow{\sigma(c)\otimes_{R_y}\sigma(b)} & S_{\gamma_2} \otimes_{R_y} S_{\eta_2} \\ \downarrow{\scriptstyle \mu_{\gamma_1,\eta_1}} & & \downarrow{\scriptstyle \mu_{\gamma_2,\eta_2}} \\ S_{\gamma_1\circ\eta_1} & \xrightarrow{\sigma(c\bullet b)} & S_{\gamma_2\circ\eta_2}. \end{array}$$

Since $c \bullet b = (c \bullet 1_{\eta_2}) \cdot (1_{\gamma_1} \bullet b)$ and $\sigma(c) \otimes_{R_y} \sigma(b) = (\sigma(c) \otimes_{R_y} 1_{S_{\eta_2}}) \cdot (1_{S_{\gamma_1}} \otimes_{R_y} \sigma(b))$, it suffices to check that this diagram commutes if $b$ or $c$ is a unit 2-arrow. Then $\sigma(b)$ or $\sigma(c)$ is the identity map. Writing the maps $\mu_{\gamma,\eta}$ multiplicatively, we get exactly the condition in the proposition. □

REMARK 4.4.5. Let $b\colon \mathrm{id}_x \Rightarrow f$ be a 2-arrow whose source is a unit arrow. Then $\sigma(b)$ is an $S_x$-bimodule map $S_x \to S_f$. Since $S_x$ is a unital ring, this map is determined by its value on the unit element $1 \in S_x$. Then it must be of the form $a \mapsto u \cdot a = a \cdot u$ for a unique element $u$ in the centre of the $S_x$-bimodule $S_f$.

If all 2-arrows in $\mathcal{C}$ are invertible, then all the bimodule maps $\sigma(b)$ in the previous proposition are bimodule isomorphisms by Exercise 4.3.4.

**4.4.2. Transformations.** Now we turn to transformations between the morphisms above. These correspond to certain graded bimodules:

DEFINITION 4.4.6. Let $\mathcal{C}$ be a category. Let $S = \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ be a $\mathcal{C}$-graded ring and let $R = \bigoplus_{x\in\mathcal{C}^0} R_x$ be a $\mathcal{C}^0$-graded ring with a nondegenerate $\mathcal{C}^0$-graded ring homomorphism $\lambda\colon R \to S|_{\mathcal{C}^0}$. A *$\mathcal{C}^0$-graded right $R$-module* is a module of the form $M := \bigoplus_{x\in\mathcal{C}^0} M_x$ where each $M_x$ is a right $R_x$-module. The *$\mathcal{C}$-graded right $S$-module induced by $M$* is

$$M \otimes_R S \cong \bigoplus_{\gamma\in\mathcal{C}} M_{r(\gamma)} \otimes_{R_{r(\gamma)}} S_\gamma \tag{4.4.1}$$

with the obvious $\mathcal{C}$-graded right $S$-module structure.

EXERCISE 4.4.7. *Prove the isomorphism asserted in* (4.4.1).

PROPOSITION 4.4.8. *Let $\mathcal{C}$ be a category. Describe two morphisms $\mathcal{C} \rightrightarrows \mathfrak{Rings}$ as in Proposition* 4.4.3 *through $\mathcal{C}$-graded rings $S = \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ and $T = \bigoplus_{\gamma\in\mathcal{C}} T_\gamma$ with nondegenerate $\mathcal{C}^0$-graded ring homomorphisms $R \to S|_{\mathcal{C}^0}$ and $U \to T|_{\mathcal{C}^0}$. A transformation between them is equivalent to an $R,U$-bimodule $M$ together with a $\mathcal{C}$-graded left $S$-module structure on the induced right $T$-module $M \otimes_U T$, which extends the canonical $R$-module structure and turns $M \otimes_U T$ into a $\mathcal{C}$-graded $S,T$-bimodule $M'$. If the target transformation is strongly unital, that is, $U \to T|_{\mathcal{C}^0}$ is an isomorphism, then the bimodule $M$ is redundant, and a transformation is equivalent to a $\mathcal{C}$-graded $S,T$-bimodule $M'$ with the property that the right multiplication map induces isomorphisms $M'_{r(\gamma)} \otimes_{T_{r(\gamma)}} T_\gamma \xrightarrow{\cong} M'_\gamma$ for all $\gamma \in \mathcal{C}$.*

*The transformation is strong if and only if the composite maps $S_\gamma \otimes_{R_{s(\gamma)}} M_{s(\gamma)} \to S_\gamma \otimes_{R_{s(\gamma)}} M_{s(\gamma)} \otimes_{U_{s(\gamma)}} T_{s(\gamma)} \to M_\gamma \otimes_{U_{s(\gamma)}} T_{s(\gamma)}$ are invertible for all $\gamma \in M$.*

PROOF. A transformation gives a $\mathcal{C}^0$-graded $R,U$-bimodule $M = \bigoplus_{x\in\mathcal{C}^0} M_x$ with $R_{r(\gamma)}, U_{s(\gamma)}$-bimodule maps $\alpha_\gamma\colon S_\gamma \otimes_{R_{s(\gamma)}} M_{s(\gamma)} \to M_{r(\gamma)} \otimes_{U_{r(\gamma)}} T_\gamma$ for all $\gamma \in \mathcal{C}$, such that the diagrams (4.3.4) and (4.3.5) commute. Let $M' := M \otimes_U T$ be the $\mathcal{C}$-graded right $T$-module induced by $M$. Let $\gamma\colon y \to z$ and $\eta\colon x \to y$ be arrows in $\mathcal{C}$. The maps $\alpha_\gamma$ induce maps

$$\alpha_{\gamma,\eta}\colon S_\gamma \otimes_{R_y} M_y \otimes_{U_y} T_\eta \xrightarrow{\alpha_\gamma \otimes_{U_y} \mathrm{id}_{T_\eta}} M_z \otimes_{U_z} T_\gamma \otimes_{U_y} T_\eta \xrightarrow{\mathrm{id}_{M_z} \otimes_{U_z} \mu_{\gamma,\eta}} M_z \otimes_{U_z} T_{\gamma\eta},$$

where $\mu_{\gamma,\eta}$ denotes the multiplication map $T_\gamma \otimes_{U_y} T_\eta \to T_{\gamma\eta}$. Conversely, $\alpha_{\gamma,1_y}$ composed with the map induced by $\lambda_y\colon U_y \to T_y$ gives back $\alpha_\gamma$. The maps $\alpha_{\gamma,\eta}$ combine to a grading-preserving map

$$S \otimes_R M' = S \otimes_R M \otimes_U T \to M \otimes_U T = M'.$$

This is a right $T$-module map because $T$ is associative. It defines a left module structure if (4.3.4) commutes, and vice versa. And (4.3.5) commutes if and only if the left $S$-module structure on $M'$ restricts to the given $R$-module structure. This implies the desired description of transformations. The criterion for a transformation to be strong follows easily from these computations.

Now assume $U \to T|_{\mathcal{C}^0}$ to be invertible. Then $M'_x = M_x \otimes_{U_x} T_x \cong M_x$ for all $x \in \mathcal{C}^0$ by the multiplication isomorphism. Thus the $\mathcal{C}$-graded $S,T$-bimodule $M'$ pins down the $\mathcal{C}^0$-graded $R,U$-bimodule $M$. And a $\mathcal{C}$-graded $S,T$-bimodule $M'$ comes from a transformation if and only if the multiplication maps $M'_{r(\gamma)} \otimes_{T_{r(\gamma)}} T_\gamma \to M'_\gamma$ are isomorphisms for all $\gamma$. □

Now let $\mathcal{C}$ be a 2-category. Two morphisms $\mathcal{C} \rightrightarrows \mathfrak{Rings}$ are described through the data in Proposition 4.4.8 together with maps $\sigma(b)\colon S_\gamma \to S_\eta$ and $\tau(b)\colon T_\gamma \to T_\eta$ for all 2-arrows $b\colon \gamma \Rightarrow \eta$ that satisfy the conditions in Proposition 4.4.4. A transformation between these two morphisms on $\mathcal{C}$ is also a transformation between the restrictions to $\mathcal{C}^{\le 1}$. So it has the same data as in Proposition 4.4.8. In addition, the $R,U$-bimodule maps $S_\gamma \otimes_{R_{s(\gamma)}} M_{s(\gamma)} \to M_{r(\gamma)} \otimes_{R_{r(\gamma)}} T_\gamma$ that give the left module structure on $M'$ are required to be natural for 2-arrows. This says that for all parallel arrows $\gamma, \eta\colon x \rightrightarrows y$ and 2-arrows $b\colon \gamma \Rightarrow \eta$, the following diagram commutes:

$$\begin{array}{ccc} S_\gamma \otimes_{R_x} M_x & \longrightarrow & M_y \otimes_{R_y} T_\gamma \\ \downarrow{\scriptstyle \sigma(b)\otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\tau(b)} \\ S_\eta \otimes_R M_x & \longrightarrow & M_y \otimes_{R_y} T_\eta \end{array}$$

**4.4.3. Modifications and icons.** The following two exercises describe modifications and icons explicitly, based on the descriptions of morphisms and transformations above.

EXERCISE 4.4.9. *Describe two morphisms $\mathcal{C} \to \mathfrak{Rings}$ as in Proposition 4.4.3 through $S = \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ with $R \to S|_{\mathcal{C}^0}$ and $T = \bigoplus_{\gamma\in\mathcal{C}} T_\gamma$ with $U \to T|_{\mathcal{C}^0}$. Describe two transformations between these morphisms as in Proposition 4.4.8 through $\mathcal{C}^0$-graded $R,U$-bimodules $M$ and $N$ with suitable $\mathcal{C}$-graded $S,T$-bimodule structures on $M \otimes_U T$ and $N \otimes_U T$. A modification between these transformations is equivalent to an $R,U$-bimodule map $\varphi\colon M \to N$ such that the induced grading-preserving right $T$-module map $\varphi \otimes_U \mathrm{id}_T\colon M \otimes_U T \to N \otimes_U T$ is also a left $S$-module map. If $U \to T|_{\mathcal{C}^0}$ is an isomorphism, then modifications are equivalent to grading-preserving $S,T$-bimodule maps $M \otimes_U T \to N \otimes_U T$.*

EXERCISE 4.4.10. *Describe two morphisms $\mathcal{C} \to \mathfrak{Rings}$ as in Proposition 4.4.3 through $S = \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ with $\iota_R\colon R \to S|_{\mathcal{C}^0}$ and $T = \bigoplus_{\gamma\in\mathcal{C}} T_\gamma$ with $\iota_U\colon U \to T|_{\mathcal{C}^0}$. Identify icons between these morphisms with $\mathcal{C}$-grading preserving ring homomorphisms $\alpha\colon S \to T$ that satisfy $\alpha \circ \iota_R = \iota_U$.*

## 4.5. Products of modifications and transformations

We have seen in Theorem 2.4.5 and Proposition 2.7.7 that twisted actions of groups and crossed modules on C*-algebras form 2-categories themselves, with transformations as arrows and modifications as 2-arrows. Similarly, we are going to build a bicategory $\mathrm{Mor}(\mathcal{C}, \mathcal{D})$ that has morphisms $\mathcal{C} \to \mathcal{D}$ as objects, transformations as arrows, and modifications as 2-arrows. It remains to define the vertical product of modifications, the product of transformations, the horizontal product of modifications, and associators and uniters for the product of transformations. These will satisfy the coherence conditions for a bicategory by construction. Since products of strong transformations remain strong, there is a subbicategory $\mathrm{Hom}(\mathcal{C},\mathcal{D})$ that has homomorphisms $\mathcal{C}\to\mathcal{D}$ as objects, strong transformations as arrows, and modifications as 2-arrows.

Morphisms, transformations, and modifications carry even more algebraic structure. Just as categories form a bicategory $\mathfrak{Cat}$, so bicategories form a "tricategory" with bicategories as objects, morphisms as arrows, transformations as 2-arrows and modifications as 3-arrows. Roughly speaking, the $j$-arrows in a tricategory for $j=1,2,3$ carry $j$ different products, which generalise the product of arrows and the horizontal and vertical products of 2-arrows in a bicategory. We will introduce more of this structure later when we need it.

LEMMA 4.5.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories, let $F,G\colon\mathcal{C}\rightrightarrows\mathcal{D}$ be morphisms, and let $\sigma,\sigma',\sigma''\colon F\Rightarrow G$ be three transformations. Let $\Gamma\colon\sigma\Rrightarrow\sigma'$ and $\Delta\colon\sigma'\Rrightarrow\sigma''$ be modifications. These are given by 2-arrows $\Gamma_x\colon\sigma_x\Rightarrow\sigma'_x$ and $\Delta_x\colon\sigma'_x\Rightarrow\sigma''_x$ for $x\in\mathcal{C}^0$. Their vertical products $\Delta_x\cdot\Gamma_x$ form a modification $\Delta\cdot\Gamma\colon\sigma\Rrightarrow\sigma''$. The transformations $F\Rightarrow G$ and the modifications between them with this product form a category. The unit arrow on $\sigma\colon F\Rrightarrow G$ is the family of unit arrows $1_{\sigma_x}\colon\sigma_x\Rightarrow\sigma_x$.*

PROOF. All claims are shown by very short computations. □

Next let $\mathcal{C}$ and $\mathcal{D}$ be bicategories, let $F,G,H\colon\mathcal{C}\rightrightarrows\mathcal{D}$ be morphisms, and let $\sigma\colon F\Rightarrow G$ and $\sigma'\colon G\Rightarrow H$ be transformations. We are going to define a transformation $\sigma'\circ\sigma\colon F\Rightarrow H$. The transformations $\sigma$ and $\sigma'$ consist of arrows $\sigma_x\colon F^0(x)\to G^0(x)$ for all $x\in\mathcal{C}^0$ and $\sigma'_x\colon G^0(x)\to H^0(x)$ for all $x\in\mathcal{C}^0$ and natural 2-arrows $\sigma_f\colon G(f)\circ\sigma_x\Rightarrow\sigma_y\circ F(f)$ and $\sigma'_f\colon H(f)\circ\sigma_x\Rightarrow\sigma_y\circ G(f)$ for all arrows $f\colon x\to y$ in $\mathcal{C}$, such that the diagrams (4.3.4) and (4.3.5) commute. Define

$$(\sigma'\circ\sigma)_x := \sigma'_x\circ\sigma_x\colon F^0(x)\to G^0(x)\to H^0(x)$$

for all $x\in\mathcal{C}$. For an arrow $f\colon x\to y$ in $\mathcal{C}$, let $(\sigma'\circ\sigma)_f$ be the vertical product

$$H(f)\circ(\sigma'_x\circ\sigma_x)\Rightarrow(H(f)\circ\sigma'_x)\circ\sigma_x\xrightarrow{\sigma'_f\bullet 1_{\sigma_x}}(\sigma'_y\circ G(f))\circ\sigma_x$$
$$\Rightarrow\sigma'_y\circ(G(f)\circ\sigma_x)\xrightarrow{1_{\sigma'_y}\bullet\sigma_f}\sigma'_y\circ(\sigma_y\circ F(f))\Rightarrow(\sigma'_y\circ\sigma_y)\circ F(f).$$

Here the unlabelled 2-arrows are associators in $\mathcal{D}$ or their inverses.

LEMMA 4.5.2. *This data defines a transformation $\sigma'\circ\sigma\colon F\Rightarrow H$.*

PROOF. We must show that the diagrams (4.3.4) and (4.3.5) commute for $\sigma'\circ\sigma$. If the target bicategory $\mathcal{D}$ is strict, then the diagram (4.3.4) commutes for the same reason as in the proof of Theorem 2.4.5. The vertices in the big diagram in that proof are products of three or four arrows. When $\mathcal{D}$ is a general bicategory, then each product of three arrows is replaced by two arrows that are linked by an invertible 2-arrow, the associator between them; and each product of four arrows is replaced by a commuting pentagon of invertible 2-arrows as in (4.2.4); this pentagon contains the five ways to put parentheses in a product of four arrows and all asociators that link them. The 2-arrows in the big diagram in the proof of Theorem 2.4.5 give 2-arrows that link suitable vertices in these pairs or pentagons. The naturality of the associators implies that the squares that compare different 2-arrows between the same associator pair or pentagon commute. The compatibility of horizontal and vertical products gives one more commuting square, and the diagrams (4.3.4) for $\sigma$ and $\sigma'$ give two more commuting polygons. This information suffices to deduce that the diagram (4.3.4) for $\sigma' \circ \sigma$ commutes. The complete diagram is in Figure 1; the symbol $\circ$ for products of arrows is left out in the diagram to save space. (See also [**21**] for this computation.) The diagram (4.3.5) for $\sigma' \circ \sigma$ is exactly the outer diagram in Figure 2. The squares and parallelograms in Figure 2 commute because associators are natural. The triangles commute because of the commuting diagrams (4.2.1) and (4.2.5). And the two pentagons – one has a bent edge to make the whole diagram smaller – commute because of the commuting diagrams (4.3.5) for $\sigma'$ and $\sigma$. Thus $\sigma' \circ \sigma$ is a transformation. By construction, it is strong if both $\sigma'$ and $\sigma$ are strong.

The naturality of the 2-arrows $(\sigma' \circ \sigma)_f$ is easy to see. For transformations between twisted actions of crossed modules, this is asserted in Proposition 2.7.7. Actually, this special case is more confusing because in a crossed module, we only keep track of certain 2-arrows, and this restriction does not go so well with composition. □

EXERCISE 4.5.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism. Describe a "unit transformation" $F \Rightarrow F$ (see also* [**13**, 4.2.12]*). Show that it acts as a unit for the composition of transformations.*

The horizontal product of 2-arrows in $\mathcal{D}$ yields a horizontal product of modifications: if $\Gamma\colon \sigma_1' \Rrightarrow \sigma_2'$ and $\Delta\colon \sigma_1 \Rrightarrow \sigma_2$ are modifications, then $(\Gamma \bullet \Delta)_x := \Gamma_x \bullet \Delta_x$ is a modification $\sigma_1' \circ \sigma_1 \Rrightarrow \sigma_2' \circ \sigma_2$; this follows using that associators are natural and that $\Gamma$ and $\Delta$ are modifications. It is left to the reader to draw the commuting diagrams of 2-arrows for this.

Let $F$ be a morphism. There is a *unit icon* $1_F\colon F \Rightarrow F$, consisting of the unit 2-arrows $1_{F(f)}\colon F(f) \Rightarrow F(f)$ for all $f \in \mathcal{C}$. It corresponds to a *unit transformation* $1_F\colon F \Rightarrow F$ on $F$ by Proposition 4.3.13. The latter consists of the unit arrows $1_{F^0(x)}\colon F^0(x) \to F^0(x)$ for $x \in \mathcal{C}^0$ and the 2-arrows

$$F(f) \circ 1_{F^0(x)} \xRightarrow[\cong]{r^{\mathcal{D}}_{F(f)}} F(f) \xRightarrow[\cong]{(l^{\mathcal{D}}_{F(f)})^{-1}} 1_{F^0(y)} \circ F(f).$$

These 2-arrows are natural because the uniters are natural. The diagram (4.3.4) commutes for $1_F$ because of the diagrams (4.2.5), and (4.3.5) commutes because the two diagrams in (4.3.2) commute. The transformation $1_F$ is always strong, and it is strict if $\mathcal{D}$ is strictly unital.

EXERCISE 4.5.4. *Let $\sigma\colon F \Rightarrow G$ be a transformation. Then the family of uniters $\sigma_x \circ 1_{F^0(x)} \Rightarrow \sigma_x$ in $\mathcal{D}$ is an invertible modification $\sigma \circ 1_F \Rrightarrow \sigma$ and the family of uniters $1_{G^0(x)} \circ \sigma_x \Rightarrow \sigma_x$ in $\mathcal{D}$ is an invertible modification $1_G \circ \sigma \Rrightarrow \sigma$.*

Let $F, G, H, K\colon \mathcal{C} \to \mathcal{D}$ be morphisms and let $\sigma\colon F \Rightarrow G$, $\sigma'\colon G \Rightarrow H$ and $\sigma''\colon H \Rightarrow K$ be transformations. We have defined transformations $(\sigma'' \circ \sigma') \circ \sigma$ and $\sigma'' \circ (\sigma' \circ \sigma)$ from $F$ to $K$. The associators $(\sigma''_x \circ \sigma'_x) \circ \sigma_x \Rightarrow \sigma''_x \circ (\sigma'_x \circ \sigma_x)$ in $\mathcal{D}$ for $x \in \mathcal{C}^0$ combine to a modification

$$(\sigma'' \circ \sigma') \circ \sigma \Rrightarrow \sigma'' \circ (\sigma' \circ \sigma).$$

It is a routine exercise to check that the diagram in Definition 4.3.10 commutes for these associators.

PROPOSITION 4.5.5. *The data above defines a bicategory, which we denote by* $\mathrm{Mor}(\mathcal{C}, \mathcal{D})$*. It is strict if $\mathcal{D}$ is strict. There is a subbicategory* $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ *that has homomorphisms $\mathcal{C} \to \mathcal{D}$ as objects, strong transformations between them as arrows, and modifications between these as* 2*-arrows.*

PROOF. The uniter and associator modifications are simply families of uniters and associators in $\mathcal{D}$. Hence they inherit the coherence conditions in the definition of a bicategory from $\mathcal{D}$. And strictness is also inherited from $\mathcal{D}$. □

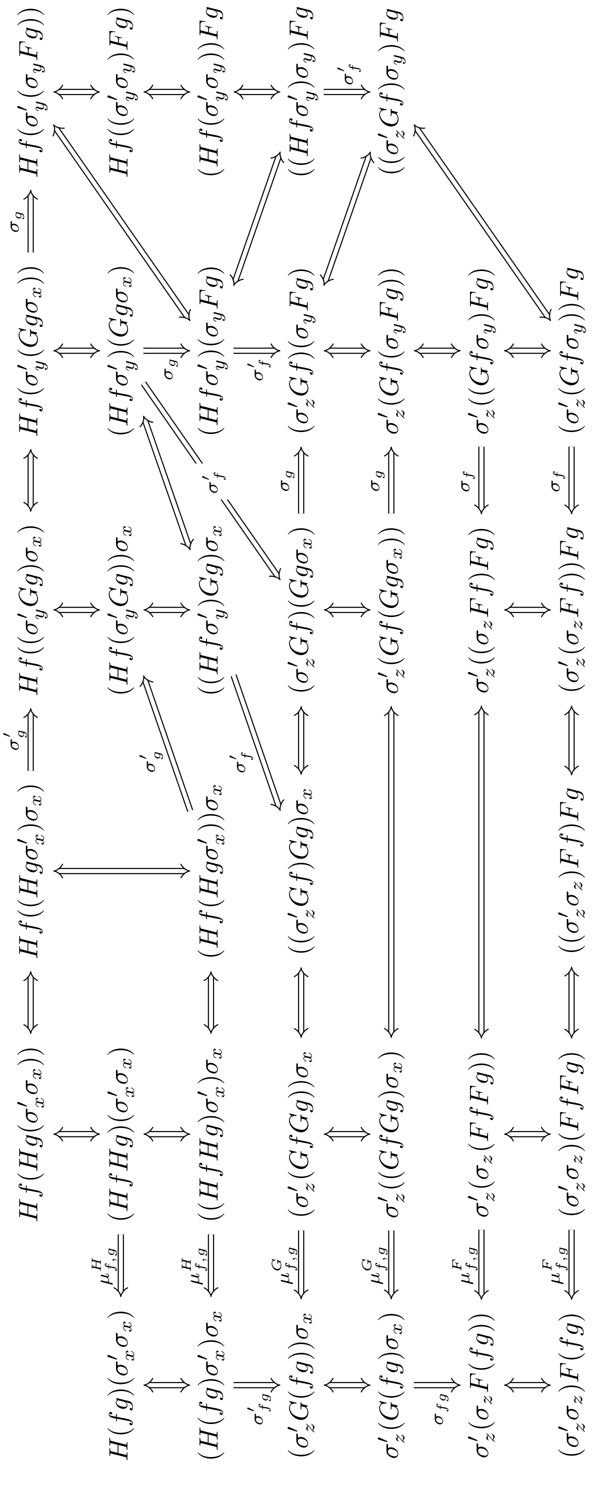


FIGURE 1. Proof that the diagram (4.3.4) commutes for $\sigma' \circ \sigma$. The double-headed arrows are horizontal products with associators, the others are horizontal products with the 2-arrows by which they are labelled.

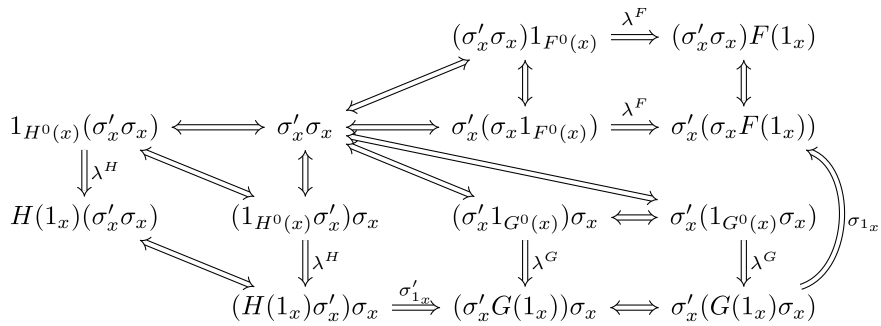


FIGURE 2. Proof that the diagram (4.3.5) commutes for $\sigma' \circ \sigma$. The double-headed arrows are horizontal products with uniters and associators. The others are horizontal products with the 2-arrows by which they are labelled.

EXERCISE 4.5.6. *Let $\mathcal{C}$ be a crossed module, viewed as a strict bicategory as in Example* 4.2.2*, and let $\mathcal{D} = \mathcal{C}^*(2)$. The definitions above make* $\mathrm{Hom}(\mathcal{C}, \mathcal{C}^*(2))$ *a* 2*-category because $\mathcal{C}^*(2)$ is a strict bicategory. Restrict further to the subbicategory* $\mathrm{Hom}_1(\mathcal{C}, \mathcal{C}^*(2))$ *with strictly unital homomorphisms as objects, and strong transformations as arrows and modifications as* 2*-arrows. The objects, arrows and* 2*-arrows in* $\mathrm{Hom}_1(\mathcal{C}, \mathcal{C}^*(2))$ *have been identified with twisted actions of $\mathcal{C}$ on* C*-*algebras, and with the transformations and modifications between these as defined in Section* 2.7 *(see Exercise* 4.3.3*, Exercise* 4.3.8 *and Exercise* 4.3.11*). Identify the* 2*-category structure on* $\mathrm{Hom}(\mathcal{C}, \mathcal{C}^*(2))$ *with the* 2*-category structure in Proposition* 2.7.7*. As a result, Proposition* 4.5.5 *generalises Proposition* 2.7.7 *and Theorem* 2.4.5 *about transformations and modifications between twisted actions of groups and crossed modules.*

**4.5.1. Composition of modifications and transformations when the target bicategory is $\mathfrak{Rings}$.** We now specialise to morphisms from a category $\mathcal{C}$ to the bicategory $\mathfrak{Rings}$. First we describe the product of transformations in terms of the bimodules in Proposition 4.4.8. Describe three morphisms $F_j\colon \mathcal{C} \rightrightarrows \mathfrak{Rings}$ as in Proposition 4.4.3 through $\mathcal{C}^0$-graded rings $R_j = \bigoplus_{x\in\mathcal{C}^0} R_{j,x}$ and $\mathcal{C}$-graded rings $S_j = \bigoplus_{\gamma\in\mathcal{C}} S_{j,\gamma}$ with nondegenerate homomorphisms $R_j \to S_j|_{\mathcal{C}^0}$ for $j = 1,2,3$. Then describe transformations $\sigma_{j,k}\colon F_k \Rightarrow F_j$ for $(j,k) = (1,2)$ and $(j,k) = (2,3)$ as in Proposition 4.4.8 through an $R_j, R_k$-bimodule $M_{j,k}$ and a $\mathcal{C}$-graded $S_j, S_k$-bimodule structure on the induced right $S_k$-module $M'_{j,k} := M_{j,k} \otimes_{R_k} S_k$ that extends the canonical $\mathcal{C}$-graded $R_j, S_k$-bimodule structure. Then $M_{1,3} := M_{1,2} \otimes_{R_2} M_{2,3}$ is an $R_1, R_3$-bimodule and there are $\mathcal{C}$-graded $R_j, S_k$-bimodule isomorphisms

$$\begin{aligned} M_{1,3} \otimes_{R_3} S_3 := M_{1,2} \otimes_{R_2} M_{2,3} \otimes_{R_3} S_3 := M_{1,2} \otimes_{R_2} M'_{2,3} \\ \cong M_{1,2} \otimes_{R_2} S_2 \otimes_{S_2} M'_{2,3} := M'_{1,2} \otimes_{S_2} M'_{2,3}. \end{aligned}$$

Here we have used that $M'_{2,3}$ is a $\mathcal{C}$-graded $S_2, S_3$-bimodule. In addition, since $M'_{1,2}$ is a $\mathcal{C}$-graded $S_1, S_2$-bimodule, the $\mathcal{C}$-graded $R_j, S_k$-bimodule structure on $M'_{1,2} \otimes_{S_2} M'_{2,3}$ extends canonically to a $\mathcal{C}$-graded $S_1, S_3$-bimodule structure. This corresponds to a transformation $F_1 \Rightarrow F_3$ by Proposition 4.4.8.

EXERCISE 4.5.7. *The transformation $F_1 \Rightarrow F_3$ that corresponds to the pair of graded tensor product bimodules $M_{1,2} \otimes_{R_2} M_{2,3}$ and $M'_{1,2} \otimes_{S_2} M'_{2,3}$ is the transformation $\sigma_{1,2} \circ \sigma_{2,3}$. Put in a nutshell, the product of transformations corresponds to the tensor product of bimodules.*

Now let the morphisms $F_j\colon \mathcal{C} \to \mathfrak{Rings}$ for $j = 1, 2, 3$ be strictly unital. Then the descriptions of morphisms and transformations in Proposition 4.4.3 and Proposition 4.4.8 simplify because the $\mathcal{C}^0$-graded rings and bimodules above become redundant. So the product of transformations is simply the tensor product of the $\mathcal{C}$-graded bimodules over the $\mathcal{C}$-graded rings that describe the transformations between strictly unital morphisms $\mathcal{C} \to \mathfrak{Rings}$.

Nothing much changes if the category $\mathcal{C}$ is replaced by a 2-category. This is because the 2-arrows in $\mathcal{C}$ give extra conditions but not extra data for transformations. And the product of transformations is compatible with forgetting the 2-arrows in $\mathcal{C}$ and treating a transformation of morphisms $\mathcal{C} \to \mathfrak{Rings}$ as a transformation between the restricted morphisms $\mathcal{C}^{\leq 1} \to \mathfrak{Rings}$. Thus the product of transformations becomes the bimodule tensor product also for morphisms from a 2-category to $\mathfrak{Rings}$.

We have not spoken yet about the products on modifications. Modifications are described in Exercise 4.4.9 through those $\mathcal{C}^0$-graded bimodule maps that induce $\mathcal{C}$-graded bimodule maps. It is evident from the solution of this exercise that the vertical products of modifications translates to the usual composition of these bimodule maps. And the horizontal product is easily described in the setting above as the tensor product over $R_2$ of an $R_1, R_2$-bimodule map and an $R_2, R_3$-bimodule map.

Remark 4.5.8. We can now explain a technical point, namely, the word "equivalent" in Proposition 4.4.8. The set of transformations is *not* equal to the set of graded bimodules described in the statement of the proposition unless $\mathcal{C}$ is a monoid, that is, there is only one object. The problem is to combine the $R_{1,x}, R_{2,x}$-bimodules $M_x$ for $x \in \mathcal{C}^0$ into a single $\mathcal{C}^0$-graded $R_1, R_2$-bimodule $M := \bigoplus_{x\in\mathcal{C}^0} M_x$. This direct sum is unique up to a canonical isomorphism, but it is not unique. When we turn the $\mathcal{C}^0$-graded $R_1, R_2$-bimodule back into a family of $R_{1,x}, R_{2,x}$-bimodules, then we replace $M_x$ by its image in $\bigoplus_{x\in\mathcal{C}^0} M_x$. This is canonically isomorphic but not equal to $M_x$. And when we start with a $\mathcal{C}^0$-graded $R_1, R_2$-bimodule $M$, decompose it into the $M_x$, and form $\bigoplus_{x\in\mathcal{C}^0} M_x$, the result is canonically isomorphic to $M$, but need not be exactly the same as a set. The best we can say is that the category of transformations and modifications is equivalent to the appropriate category of graded bimodules. This is exactly what the proof shows and what the statement means. So Proposition 4.4.8 had actually been meaningless before we introduced the vertical product of modifications in Lemma 4.5.1.

There is a similar but more serious issue with the word "equivalent" in Proposition 4.4.3. Now the statement is that certain *bi*categories are equivalent – and we have not yet talked about that. Even worse, we have not yet defined the bicategory of graded bimodules whose equivalence to the bicategory $\mathrm{Mor}(\mathcal{C}, \mathfrak{Rings})$ is asserted in Proposition 4.4.3. An object of this bicategory is a triple $(R, S, \lambda)$, where $R$ is a $\mathcal{C}^0$-graded ring, $S$ is a $\mathcal{C}$-graded ring, and $\lambda\colon R \to S|_{\mathcal{C}^0}$ is a homomorphism of $\mathcal{C}^0$-graded rings. Its arrows are pairs $(M, \vartheta)$ where $M$ is a $\mathcal{C}^0$-graded $R, U$-bimodule and $\vartheta$ is a $\mathcal{C}$-graded left $S$-module structure on $M' := M \otimes_U T$ that makes $M'$ a $\mathcal{C}$-graded $S, T$-bimodule. Its 2-arrows are $\mathcal{C}^0$-graded $R, U$-bimodule maps that induce $\mathcal{C}$-graded $S, T$-bimodule maps. The arrows are composed through bimodule tensor products as described above, and the vertical and horizontal products of 2-arrows are composition of bimodule maps and the tensor product of bimodule maps. It is easy to check that this defines a bicategory. The precise meaning of Proposition 4.4.3 is that this somewhat technical bicategory is equivalent to the bicategory $\mathrm{Mor}(\mathcal{C}, \mathfrak{Rings})$. The statement gets less technical when we restrict attention to strictly unital morphisms. This is equivalent to a subbicategory. The objects of the latter may be identified with $\mathcal{C}$-graded rings $S$, and its arrows may be identified with $\mathcal{C}$-graded $S,T$-bimodules that are induced as a right $T$-module; its 2-arrows then become just $\mathcal{C}$-graded $S,T$-bimodule homomorphisms.

So it will still take quite a bit of theory to define the word "equivalent" in Proposition 4.4.3 precisely. And the only good definition that I know will make Proposition 4.4.8 and Exercise 4.4.9 parts of the statement because an equivalence of bicategories also describes the arrows and 2-arrows, not just the objects. Most of the proof of this precise statement is done above. What is still missing is the criterion in Theorem 4.10.5 for a morphism between two bicategories to be an equivalence. This is similar to Theorem 2.8.2, where an equivalence of 2-categories is described through more concrete statements about objects, arrows and 2-arrows. The arguments above verify the criterion in Theorem 2.8.2.

Having translated the product of transformations into the language of bimodule tensor products, we may now characterise the equivalences in $\mathrm{Mor}(\mathcal{C},\mathfrak{Rings})$ in the language of graded Morita equivalence of graded rings:

Proposition 4.5.9. *Let $\mathcal{C}$ be a category and let $F,G\colon \mathcal{C}\rightrightarrows \mathfrak{Rings}$ be morphisms. Describe a transformation $\sigma\colon F\Rightarrow G$ as in Proposition 4.4.8 through a $\mathcal{C}^0$-graded $R,U$-bimodule $M$ and a $\mathcal{C}$-graded $S,T$-bimodule structure on the induced right $T$-module $M' := M\otimes_U T$ that extends the canonical $\mathcal{C}$-graded $R,T$-bimodule structure. The transformation $\sigma$ is an equivalence if and only if $M$ is a $\mathcal{C}^0$-graded Morita equivalence bimodule for $R$ and $U$ and $M'$ is a $\mathcal{C}$-graded Morita equivalence bimodule for $S$ and $T$, that is, there are a $\mathcal{C}^0$-graded $U,R$-bimodule $M^*$ with $M\otimes_U M^*\cong R$ and $M^*\otimes_R M\cong U$ and a $\mathcal{C}$-graded $T,S$-bimodule $(M')^*$ with $M'\otimes_T (M')^*\cong S$ and $(M')^*\otimes_S M'\cong T$.*

Proof. First assume that the transformation $\sigma$ is an equivalence. Then we describe the inverse transformation $\tau$ through a $\mathcal{C}^0$-graded $U,R$-bimodule $M^*$ and a $\mathcal{C}$-graded $T,S$-bimodule structure on $(M')^* := M^*\otimes_R S$ that extends the given $R,S$-bimodule structure. The transformations $\tau\circ\sigma$ and $\sigma\circ\tau$ correspond to the bimodule pairs $M^*\otimes_U M$, $(M^*)'\otimes_T M'$ and $M\otimes_R M^*$, $M'\otimes_S (M^*)'$, and the invertible modifications $\tau\circ\sigma\cong 1_F$ and $\sigma\circ\tau\cong 1_G$ correspond to bimodule isomorphisms $M^*\otimes_U M\cong R$, $(M^*)'\otimes_T M'\cong U$ and $M\otimes_R M^*\cong U$, $M'\otimes_S (M^*)'\cong T$. Thus we get the asserted graded Morita equivalences. Actually, we get a bit more because the equivalence bimodule $(M^*)'$ above is induced by $M^*$. If we assume that $M$ and $M'$ are graded Morita equivalence bimodules and that the inverse of $M'$ is induced by $M^*$, then the argument above may be reversed and shows that the transformation is an equivalence. So it remains to prove that if both $M$ and $M'$ are graded Morita equivalences, then the inverse of $M'$ must be the bimodule $(M^*)'$ induced by the inverse $M^*$ of $M$. We leave it as an exercise to check this. By the way, a bit more may be done: the inverse $M^*$ is automatically $\mathcal{C}^0$-graded if $M$ is a $\mathcal{C}^0$-graded $R,U$-bimodule and a Morita equivalence $R,U$-bimodule. □

Equivalences in the bicategory $\mathrm{Mor}(\mathcal{C},\mathfrak{Rings})$ are an appropriate analogue of the cocycle-conjugacies between twisted actions in Table 1.

## 4.6. Cones and covariance rings

There are two lines in Table 1 that we have not yet carried over to bicategories, namely, invariant maps and equalities of invariant maps, which correspond to covariant representations and the unitary intertwiners between them. Covariant representations occur in the universal property that defines the covariance algebra (or crossed product) of a twisted action of a group or crossed module on a C*-algebra. This makes them very important. There are two ways to generalise covariant representations in $\mathcal{C}^*(2)$ to general bicategories, namely, cones and lax cones. These are, at the same time, bicategorical analogues of the cones that appear in the definition of the limit of a diagram in a category (see [**23**, Section 3.1]). Calling the objects of interest cones and lax cones seems appropriate because the name "representation" no longer fits in the generality that we are considering.

We want to use lax cones and cones to define analogues of the covariance algebras in Chapter 2 for morphisms to $\mathfrak{Rings}$. There are two ways to do this. In this section, we follow the technically easier way. Namely, we single out certain classes of (lax) cones over $F$ so as to define a functor from the category of rings and ring homomorphisms to sets. The lax and strong covariance rings are defined as representing objects for these functors. This definition has two advantages. First, it pins down the lax and strong covariance rings up to isomorphism, not just up to Morita equivalence. Secondly, it uses no further bicategory theory. One disadvantage is that it needs the category of rings and homomorphisms in addition to the bicategory $\mathfrak{Rings}$. So it cannot be used in an abstract bicategory. And we cannot directly apply general results from bicategory theory to the covariance rings so defined. Since we have not yet developed any of that there, this is not much of a drawback.

Actually, the lax covariance ring for a morphism $\mathcal{C} \to \mathfrak{Rings}$ is only defined if $\mathcal{C}^0$ is finite. We need this restriction to stay within the realm of *unital* rings. Choosing the right bicategory of nonunital rings, the definition should work for morphisms defined on arbitrary bicategories. There are, however, several reasonable candidates for this larger bicategory, and I do not want to investigate this particular issue right now. Therefore, I only sketch some possibilities and issues in Section 4.6.5 and limit the discussion to unital rings and morphisms defined on bicategories with finitely many objects.

If $\mathcal{C}$ is a bicategory with finitely many objects, then any morphism $\mathcal{C} \to \mathfrak{Rings}$ has a lax covariance ring. In fact, if $\mathcal{C}$ is a category with finitely many objects, then the lax covariance ring is simply the $\mathcal{C}$-graded ring $S$ in Proposition 4.4.3. This follows easily from the description of transformations for morphisms $\mathcal{C} \to \mathfrak{Rings}$ in Section 4.4. Exercise 2.4.7 may already suggest to view such graded rings as analogues of covariance algebras for twisted group actions. It shows that the grading-preserving arrows and 2-arrows between the covariance algebras in $\mathcal{C}^*(2)$ are equivalent to transformations and morphisms. Proposition 4.4.3 makes an analogous statement in the bicategory $\mathfrak{Rings}$.

It is unclear which morphisms $F\colon \mathcal{C} \to \mathfrak{Rings}$ have a strong covariance ring. We only prove that a strong covariance ring exists if $F$ is a morphism from a bicategory with finitely many objects to the subbicategory $\mathfrak{Rings}_{\mathrm{fp}}$ that has only those bimodules as arrows that are finitely generated and projective as right modules; such bimodules already came up in the study of Morita equivalence in Theorem 4.1.13. For a morphism to $\mathfrak{Rings}_{\mathrm{fp}}$, the strong covariance ring is a Cohn localisation of the lax covariance ring.

In Section 4.6.3, we examine a concrete example of this, namely, a homomorphism $\mathbb{N} \to \mathfrak{Rings}_{\mathrm{fp}}$ that is generated by a finite directed graph. In this case, the lax covariance ring is the path algebra of the graph and the strong covariance ring is its Leavitt path algebra (see [**2**]). It is already known that the Leavitt path algebra of a graph is a Cohn localisation of the path algebra of the graph (see [**4**, **5**]). Leavitt path algebras are purely algebraic analogues of graph C*-algebras, and this has made them interesting for operator algebraists. Our discussion here is limited to finite graphs without sources.

We specialise to group actions by automorphisms in Section 4.6.4. In this case, any transformation is strong, so that the two covariance rings coincide. And the covariance ring is simply the usual crossed product for the group action. This result also remains true for twisted group actions. As a result, the Leavitt path algebra of a finite directed graph and the crossed product for a (possibly twisted) group action by automorphisms satisfy similar universal properties. This is a purely algebraic analogue of a statement in the C*-correspondence bicategory in [**3**].

We mentioned above that there is another way to define analogues of covariance algebras for morphisms to $\mathfrak{Rings}$; this definition is introduced in [**3**] for homomorphisms to the C*-correspondence bicategory. The (lax) cones over $F$ with a given ring $D$ as summit form a category, and mapping $D$ to this category is part of a homomorphism $\mathfrak{Rings} \to \mathfrak{Cat}$. The (lax) limit of $F$ is defined as an object of $\mathfrak{Rings}$ that represents this homomorphism. The lax and strong covariance rings defined above are also a lax limit and a limit in this sense, respectively. An advantage of (lax) limits is that they may be defined in arbitrary bicategories and have good naturality properties: there is a homomorphism $\mathrm{Hom}(\mathcal{C}, \mathfrak{Rings}) \to \mathfrak{Rings}$ that maps a homomorphism $\mathcal{C} \to \mathfrak{Rings}$ to its strong covariance ring. As a result, equivalent homomorphisms have equivalent strong covariance rings. We may even replace the domain bicategory $\mathcal{C}$ of a morphism to $\mathfrak{Rings}$ by an equivalent bicategory $\mathcal{C}'$. Then there is an analogue of Theorem 2.8.2: the bicategories of homomorphisms $\mathcal{C} \to \mathfrak{Rings}$ and $\mathcal{C}' \to \mathfrak{Rings}$ are equivalent if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent, and if a homomorphism $F\colon \mathcal{C} \to \mathfrak{Rings}$ and a homomorphism $F'\colon \mathcal{C}' \to \mathfrak{Rings}$ correspond to each other under this equivalence, then the strong covariance rings of $F$ and $F'$ are Morita equivalent.

The second approach requires more bicategory theory. Merely to define the limit and the lax limit, we must build the homomorphism $\mathcal{C} \to \mathfrak{Cat}$ that is represented by an object of a bicategory $\mathcal{C}$, prove an analogue of the Yoneda Lemma for homomorphisms from a bicategory to $\mathfrak{Cat}$, and define the product of morphisms between bicategories. To prove the analogue of Theorem 2.8.2, we also need the concept of equivalence of bicategories, which requires further products on transformations and modifications. Therefore, we postpone the discussion of the second approach to Section 4.7.4.

### 4.6.1. Lax cones and lax covariance rings.

DEFINITION 4.6.1. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $x$ be an object of $\mathcal{D}$. The *constant homomorphism* $\mathrm{const}_x\colon \mathcal{C} \to \mathcal{D}$ defined by $x$ is the strict homomorphism where each object, arrow and 2-arrow in the definition of a homomorphism are $x$, the identity arrow $1_x$, or the identity 2-arrow on $1_x$, respectively.

EXAMPLE 4.6.2. Let $D$ be a ring and let $\mathcal{C}$ be a category. The homomorphism $\mathrm{const}_D\colon \mathcal{C} \to \mathfrak{Rings}$ corresponds to a $\mathcal{C}$-graded ring by Proposition 4.4.3. This ring is the category ring of $\mathcal{C}$ with coefficients in $D$. Namely, it is $T := \bigoplus_{\gamma\in\mathcal{C}} D$ with the multiplication defined as follows. If $\gamma, \eta \in \mathcal{C}$, $u_1, u_2 \in D$, then $(u_1\gamma)\cdot(u_2\eta)$ is $(u_1 \cdot u_2)\,(\gamma\eta)$ if $s(\gamma) = r(\eta)$, and 0 otherwise.

DEFINITION 4.6.3. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism and let $D \in \mathcal{D}^0$. A *lax cone over $F$ with summit $D$* is a transformation $\mathrm{const}_D \Rightarrow F$. Let $\mathrm{Cone}_{\mathrm{lax}}(D, F)$ be the category that has these lax cones as objects, modifications between them as arrows, with the product of Lemma 4.5.1. A *cone over $F$ with summit $D$* is a strong transformation $\mathrm{const}_D \Rightarrow F$. Let $\mathrm{Cone}(D, F) \subseteq \mathrm{Cone}_{\mathrm{lax}}(D, F)$ be the full subcategory of cones.

Dually, a *lax cone under $F$ with nadir $D$* is a transformation $F \Rightarrow \mathrm{const}_D$, and a *cone under $F$ with nadir $D$* is a strong transformation $F \Rightarrow \mathrm{const}_D$. These are the objects of categories $\mathrm{Cone}(F, D) \subseteq \mathrm{Cone}_{\mathrm{lax}}(F, D)$.

PROPOSITION 4.6.4. *Let $\mathcal{C}$ be a category. Describe a morphism $F\colon \mathcal{C} \to \mathfrak{Rings}$ by graded rings $R = \bigoplus_{x\in\mathcal{C}^0} R_x$ and $S = \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ with a nondegenerate $\mathcal{C}^0$-graded homomorphism $\lambda\colon R \to S|_{\mathcal{C}^0}$. Let $D$ be a ring. The category* $\mathrm{Cone}_{\mathrm{lax}}(D, F)$ *is* *equivalent to the category of* nondegenerate *$S, D$-bimodules and bimodule homomorphisms. Here an $S, D$-bimodule is nondegenerate if the right $D$-module structure is unital and $S \cdot M = M$.*

Proof. A transformation $\mathrm{const}_D \Rightarrow F$ is given by $R_x, D$-bimodules $M_x$ for $x \in \mathcal{C}^0$ and $R_x, D$-bimodule maps $\sigma_\gamma\colon S_\gamma \otimes_{R_x} M_x \to M_y \otimes_D D$ for arrows $\gamma\colon x \to y$ in $\mathcal{C}$, such that the two coherence diagrams in Definition 4.3.7 commute. Since $M_y \otimes_D D \cong M_y$ as an $R_y, D$-module, we may replace $\sigma_\gamma$ by an $R_y, D$-bimodule map $S_\gamma \otimes_{R_x} M_x \to M_y$. Let $M := \bigoplus_{x \in \mathcal{C}^0} M_x$. Extend the $R_y, R_x$-bimodules $S_\gamma$ to $R$-bimodules by defining $R_{y'} \cdot S_\gamma = 0$ and $S_\gamma \cdot R_{x'} = 0$ for $y \neq y'$ and $x \neq x'$. Then $S_\gamma \otimes_R M = S_\gamma \otimes_{R_x} M_x$, and we may turn $\sigma_\gamma$ into an $R, D$-bimodule map $S_\gamma \otimes_R M \to M$ – use the projections onto $R_x$ in the multiplier ring of $R$ to see that the range of any such bimodule map is contained in $M_y$. These bimodule maps for all arrows $\gamma$ in $\mathcal{C}$ combine to an $R, D$-bimodule map $\sigma\colon S \otimes_R M \to M$.

A lax cone is a kind of transformation. So it makes the two diagrams (4.3.4) and (4.3.5) commute. The diagram in (4.3.4) says exactly that $\sigma$ is a left $S$-module structure on $M$. Then $M$ becomes an $S, D$-bimodule. It is nondegenerate by construction. The diagram in (4.3.5) says that this $S$-module structure restricted to $R$ gives the original $R$-module structure on $M$. Thus the $R$-module structure on $M$ is redundant. Conversely, let $M$ be a nondegenerate $S, D$-bimodule. Let $M_x := S_x \cdot M \subseteq M$. The canonical map $\bigoplus_{x \in \mathcal{C}^0} M_x \to M$ is injective because the subrings $S_x \subseteq S$ are orthogonal. It is surjective because

$$M = S \cdot M = \sum_{\gamma \in \mathcal{C}} S_\gamma \cdot M = \sum_{\gamma \in \mathcal{C}} S_{r(\gamma)} \cdot S_\gamma \cdot M = \sum_{x \in \mathcal{C}^0} S_x \cdot M = \sum_{x \in \mathcal{C}^0} M_x.$$

Reversing the proof above, the $S$-bimodule structure induces an $R_x, D$-bimodule structure on $M_x$ and gives bimodule homomorphisms $\sigma_\gamma\colon S_\gamma \otimes_{R_x} M_x \to M_y \otimes_D D$ for arrows $\gamma\colon x \to y$ in $\mathcal{C}$, such that the two coherence diagrams in Definition 4.3.7 commute and the $S$-module structure on $M \cong \bigoplus_{x \in \mathcal{C}^0} M_x$ built above is the given one. Thus a nondegenerate $S, D$-bimodule gives a transformation and vice versa.

A modification is a family of $R_x, D$-bimodule maps $f_x\colon M_x \to M_x$ with a coherence condition. We may combine all these maps into a single $R, D$-bimodule map $\bigoplus_{x \in \mathcal{C}^0} f_x\colon \bigoplus_{x \in \mathcal{C}^0} M_x \to \bigoplus_{x \in \mathcal{C}^0} M_x$. The coherence condition for the maps $f_x$ to form a modification says exactly that $f$ is a homomorphism for the nondegenerate left $S$-module structure constructed above. The $S$-module structure pins down the $R$-module structure, and $f$ is also required to be a right $D$-module homomorphism. Thus modifications are the same as $S, D$-bimodule maps.

The arguments above give an equivalence of categories between $\mathrm{Cone}_{\mathrm{lax}}(D, F)$ and the category of nondegenerate $S, D$-bimodules. This equivalence is not an isomorphism for the same reason as in Remark 4.5.8. □

If $\mathcal{C}^0$ is finite, then the ring $S$ in Proposition 4.6.4 is unital. Then a nondegenerate $S, D$-bimodule is the same as a "unital" $S, D$-bimodule or, in other words, an arrow $S \to D$ in the category $\mathfrak{Rings}$. In particular, this applies if $\mathcal{C}$ has only one object, that is, if $\mathcal{C}$ is a monoid. If $\mathcal{C}^0$ is infinite, however, then $S$ need not be unital. Thus it is not an object of $\mathfrak{Rings}$. To rectify this, we may enlarge $\mathfrak{Rings}$ to allow also some nonunital rings, including at least the $\mathcal{C}$-graded rings above. There are, however, several reasonable choices for the class of nonunital rings to consider, and the choice affects some aspects of the theory, even in the unital case. The details of this are not worked out, and I merely hint at some of the issues and possibilities in Section 4.6.5.

Definition 4.6.5. Let $\mathcal{C}$ be a bicategory with finite $\mathcal{C}^0$. A *lax covariant representation* of $F\colon \mathcal{C} \to \mathfrak{Rings}$ on a ring $D$ consists of a family of orthogonal idempotent elements $(p_x)_{x \in \mathcal{C}^0}$ in $D$ with $\sum_{x \in \mathcal{C}^0} p_x = 1$ and a transformation

$\mathrm{const}_D \Rightarrow F$ where $M_x = p_x \cdot D \subseteq D$ for all $x \in \mathcal{C}^0$. A *lax covariance ring* for $F$ is a ring $U$ such that ring homomorphisms $U \to D$ are naturally in bijection with lax covariant representations of $F$ on $D$. A *strong covariant representation* is a lax covariant representation with a strong transformation $\mathrm{const}_D \Rightarrow F$. The *strong covariance ring* for $F$ is defined like the lax one, but with strong instead of lax covariant representations.

EXAMPLE 4.6.6. If $\mathcal{C}$ has only one object, then the projection $p_x$ for the unique object must be 1. So a lax covariant representation is the same as a transformation $\mathrm{const}_D \Rightarrow F$ where the underlying $F(x), D$-bimodule is $D$ with some left $F(x)$-module structure, and similarly for strong covariant representations.

Besides the idempotent elements $p_x$, a lax covariant representation also gives right $D$-module homomorphisms $S_\gamma \otimes (p_{s(\gamma)} \cdot D) \cong p_{r(\gamma)} \cdot D$ for all $\gamma \in \mathcal{C}$. If $f\colon D \to E$ is a ring homomorphism, then a lax covariant representation on $D$ induces one on $E$ by taking the projections $f(p_x)$ and identifying the induced $E$-module homomorphism $S_\gamma \otimes_{S_{s(\gamma)}} (p_{s(\gamma)} \cdot D) \otimes_D E \to p_{r(\gamma)} \cdot D \otimes_D E$ with a homomorphism $S_\gamma \otimes_{S_{s(\gamma)}} (f(p_{s(\gamma)}) \cdot E) \to f(p_{r(\gamma)}) \cdot E$; here we have used Exercise 4.1.7 and the additivity of tensor products. So lax covariant representations are covariantly functorial in the target ring $D$. This functoriality is implicitly used in Definition 4.6.5 to define the lax covariance ring. By the Yoneda Lemma, the covariance ring is unique up to a unique ring isomorphism.

PROPOSITION 4.6.7. *Let $\mathcal{C}$ be a category with finite $\mathcal{C}^0$. Describe a morphism $F\colon \mathcal{C} \to \mathfrak{Rings}$ by graded rings $R = \bigoplus_{x \in \mathcal{C}^0} R_x$ and $S = \bigoplus_{\gamma \in \mathcal{C}} S_\gamma$ with a $\mathcal{C}^0$-graded ring homomorphism $\lambda\colon R \to S|_{\mathcal{C}^0}$. Then $S$ is a lax covariance ring of $F$.*

PROOF. The unit elements $1 \in S_x \subseteq S$ for $x \in \mathcal{C}^0$ form a family of idempotent elements $p_x \in S$ with $\sum_{x \in \mathcal{C}^0} p_x = 1$. The proof of Proposition 4.6.4 gives a bijection between lax covariant representation of $F$ on $D$ and $S, D$-bimodules. It is an exercise to check that these bijections for different rings $D$ are natural in the formal sense. □

PROPOSITION 4.6.8. *Let $\mathcal{C}$ be a strict 2-category with finitely many objects. Let $\mathcal{C}^{\le 1}$ be the underlying category of objects and arrows. Describe a morphism $F\colon \mathcal{C} \to \mathfrak{Rings}$ as in Proposition 4.4.4, through a $\mathcal{C}^0$-graded ring $R = \bigoplus_{x \in \mathcal{C}^0} R_x$, a $\mathcal{C}^{\le 1}$-graded ring $S = \bigoplus_{\gamma \in \mathcal{C}^{\le 1}} S_\gamma$ with a $\mathcal{C}^0$-graded ring homomorphism $\lambda\colon R \to S|_{\mathcal{C}^0}$, and a family of $S_{r(\gamma)}, S_{s(\gamma)}$-bimodule homomorphisms $\sigma(b)\colon S_\gamma \to S_\eta$ for all 2-arrows $b\colon \gamma \Rightarrow \eta$ in $\mathcal{C}$, with certain properties. For a 2-arrow $b\colon \gamma \Rightarrow \eta$ in $\mathcal{C}$, let*

$$J_b := \{\sigma(b)(s) - s : s \in S_\gamma\}.$$

*Then $J := \sum_b J_b$ is a two-sided ideal in $S$. And $S/J$ is the lax covariance ring for the morphism $F\colon \mathcal{C} \to \mathfrak{Rings}$.*

PROOF. Let $x, y, z \in \mathcal{C}^0$, $\gamma, \eta \in \mathcal{C}(x, y)$, $\xi \in \mathcal{C}(y, z)$, and let $b\colon \gamma \Rightarrow \eta$ be a 2-arrow in $\mathcal{C}$. Let $s \in S_\gamma$, $t \in S_\xi$. Then $t \cdot s \in S_\xi \cdot S_\gamma = S_{\xi \circ \gamma}$, and Proposition 4.4.4 says that $\sigma(1_\xi \bullet b)(t \cdot s) = t \cdot \sigma(b)(s)$. Then

$$t \cdot (\sigma(b)(s) - s) = \sigma(1_\xi \bullet b)(t \cdot s) - t \cdot s \in J_{1_\xi \bullet b} \subseteq J.$$

If $\xi \in \mathcal{C}(y', z)$ with $y' \neq y$ and $t \in S_\xi$, then $t \cdot (\sigma(b)(s) - s) = 0 \in J$ as well. This shows that $J$ is a left ideal because $S = \bigoplus_{\xi \in \mathcal{C}} S_\xi$. A similar computation shows that $J$ is a right ideal. Then the quotient $S/J$ is again a ring. Let $D$ be another ring. An $S/J, D$-bimodule is the same as an $S, D$-bimodule $M$ with the extra property that $\sigma(b)(s) \cdot m = s \cdot m$ for all 2-arrows $b\colon \gamma \Rightarrow \eta$ in $\mathcal{C}$, $s \in S_\gamma$, $m \in M$. We interpret the $S, D$-bimodule $M$ as a transformation from $\mathrm{const}_D$ to the restricted morphism $F^{\le 1}\colon \mathcal{C}^{\le 1} \to \mathfrak{Rings}$. as in Proposition 4.6.4. A transformation from $\mathrm{const}_D$ to $F\colon \mathcal{C} \to \mathfrak{Rings}$ has the same data as a transformation from $\mathrm{const}_D$ to $F^{\le 1}$, but is subject to an extra naturality condition for the 2-arrows in $\mathcal{C}$. This condition holds if and only if the left $S$-module structure on $M$ comes from an $S/J$-module structure. □

EXERCISE 4.6.9. *Let $\mathcal{C}$ be a bicategory with finitely many objects. Show that its lax covariance ring exists.*

**4.6.2. Strong covariance rings.** We have already defined strong covariant representations and the strong covariance ring along with lax covariant representations and the lax covariance ring. Now we study strong covariance rings. The lax covariance ring of a morphism is a rather obvious construction. Given a morphism to $\mathfrak{Rings}$, we merely assemble all the bimodules and bimodule maps in the morphism into a graded ring and then forget the grading. In contrast, we usually cannot read off the strong covariance ring directly from the data of the morphism. In fact, it is unclear in which generality a strong covariance ring exists. We begin with a case where there is nothing to do:

EXAMPLE 4.6.10. If all arrows in the domain bicategory $\mathcal{C}$ are equivalences and $F\colon \mathcal{C} \to \mathfrak{Rings}$ is a homomorphism, then any lax covariant representation is strong by Proposition 4.3.9. Thus the lax covariance ring is also a strong covariance ring.

Let $\mathfrak{Rings}_{\mathrm{fp}} \subseteq \mathfrak{Rings}$ be the subbicategory whose arrows are only those bimodules that are finitely generated and projective as right modules. Let $\mathcal{C}$ be a category with only finitely many objects and let $F\colon \mathcal{C} \to \mathfrak{Rings}_{\mathrm{fp}}$ be a morphism. Describe $F$ through a $\mathcal{C}^0$-graded ring $R$ and a $\mathcal{C}$-graded ring $S$ with a $\mathcal{C}^0$-graded ring homomorphism $R \to S|_{\mathcal{C}^0}$ as in Proposition 4.4.3. Let $D$ be another ring. A strong covariant representation of $F$ on $D$ is also a lax covariant representation. Thus it generates an $S,D$-bimodule $M$ by Proposition 4.6.4. Therefore, the strong covariance ring must have the property that its category of left modules is contained in the category of left $S$-modules.

Since $F$ is a morphism to $\mathfrak{Rings}_{\mathrm{fp}}$, each $S_g$ is finitely generated and projective as a right $S_{s(g)}$-module. Then both $S_g \otimes_{S_{s(g)}} S$ and $S_{r(g)} \otimes_{S_{r(g)}} S$ are finitely generated and projective right $S$-modules. We identify

$$S_g \otimes_{S_{s(g)}} S \cong \bigoplus_{h\in r^{-1}(s(g))} S_g \otimes_{S_{s(g)}} S_h, \qquad S_{r(g)} \otimes_{S_{r(g)}} S \cong \bigoplus_{k\in r^{-1}(r(g))} S_k \subseteq S.$$

The multiplication maps $\mu_{g,h}\colon S_g \otimes_{S_{s(g)}} S_h \to S_{gh}$ give a right $S$-module map

$$\psi_g\colon S_g \otimes_{S_{s(g)}} S \to S_{r(g)} \otimes_{S_{r(g)}} S.$$

Let $M$ be an $S,D$-bimodule. The left $S$-action on $M$ implies a direct sum decomposition $M = \bigoplus_{x\in\mathcal{C}^0} M_x$ – this is, in fact, how $M$ was built in the proof of Proposition 4.6.4. The map $\psi_g$ induces a right $D$-module map

$$\psi_g \otimes_S \mathrm{id}_M\colon S_g \otimes_{S_{s(g)}} M_{s(g)} \cong S_g \otimes_{S_{s(g)}} M \to S_{r(g)} \otimes_{S_{r(g)}} M = M_{r(g)}. \tag{4.6.1}$$

By definition, the transformation that belongs to $M$ by Proposition 4.6.4 is strong if and only if the maps $\psi_g$ are invertible for all $g \in \mathcal{C}^0$.

DEFINITION 4.6.11. Let $R$ be a ring. Let $u_i\colon P_i \to Q_i$ for $i \in I$ be a set of right $R$-module maps between finitely generated, projective right $R$-modules $P_i$ and $Q_i$. The *Cohn localisation* of $R$ at the set $\{u_i : i \in I\}$ is the universal ring $R'$ with a homomorphism $R \to R'$ such that the maps $u_i \otimes_R R'\colon P_i \otimes_R R' \to Q_i \otimes_R R'$ are invertible for all $i \in I$. That is, if $D$ is another ring and $f\colon R \to D$ is a homomorphism, then $f$ factors through $R'$ if and only if $u_i \otimes_R D\colon P_i \otimes_R D \to Q_i \otimes_R D$ is invertible for all $i \in I$, and this factorisation is unique if it exists.

LEMMA 4.6.12. *Let $R$ be a ring. Let $u_i\colon P_i \to Q_i$ for $i \in I$ be a set of right $R$-module maps between finitely generated, projective right $R$-modules $P_i$ and $Q_i$. Then the Cohn localisation of $R$ at the maps $u_i$ for $i \in I$ exists.*

The proof of the lemma describes the Cohn localisation more explicitly. The proof uses that all the modules $P_i$ and $Q_i$ are finitely generated and projective.

PROOF. If $i \in I$, then there are $n_i \in \mathbb{N}$, idempotent matrices $p_i, q_i \in \mathbb{M}_{n_i}(R)$, and right $R$-module homomorphisms $P_i \cong p_i \cdot R^{n_i}$ and $Q_i \cong q_i \cdot R^{n_i}$. Use these isomorphisms to transfer $u_i$ to a right $R$-module homomorphism $u_i'\colon p_i \cdot R^{n_i} \to q_i \cdot R^{n_i}$. There is a unique matrix $m_i \in q_i \cdot \mathbb{M}_{n_i}(R) \cdot p_i$ such that $u_i'(x) = m_i \cdot x$ for all $x \in p_i \cdot R^{n_i} \subseteq R^{n_i}$. Let $S$ be the ring obtained from $R$ by adjoining elements $(m_i^*)_{j,k}$ for $i \in I$, $1 \le j, k \le n_i$, subject to the relations

$$p_i \cdot m_i^* = m_i^* = m_i^* \cdot q_i, \qquad m_i^* \cdot m_i = p_i, \qquad m_i \cdot m_i^* = q_i.$$

These matrix equations say that two matrices have the same entries. For instance, the third relation says that $\sum_{k=1}^{n_i} (m_i)_{j,k}(m_i^*)_{k,l} = (q_i)_{j,l}$ for all $1 \le j, l \le n_i$. The relations say that $m_i^*$ is equivalent to the map $q_i \cdot S^{n_i} \to p_i \cdot S^{n_i}$, $x \mapsto m_i^* \cdot x$, and that this map is inverse to the map $p_i \cdot S^{n_i} \to q_i \cdot S^{n_i}$, $x \mapsto m_i \cdot x$. Therefore, $u_i \otimes_R S\colon P_i \otimes_R S \to Q_i \otimes_R S$ is invertible for all $i \in I$. Let $f\colon R \to D$ be any ring homomorphism with the property that $u_i \otimes_R D$ is invertible for all $i \in I$. Then the induced homomorphism $f_* = \mathbb{M}_{n_i}(f)\colon \mathbb{M}_{n_i}(R) \to \mathbb{M}_{n_i}(D)$ maps $m_i \in q_i \mathbb{M}_{n_i}(R) p_i$ to an invertible element in $f_*(q_i)\mathbb{M}_{n_i}(D) f_*(p_i)$. Map the extra generators $(m_i^*)_{j,k}$ of $S$ to the entries of the matrix in $f_*(p_i)\mathbb{M}_{n_i}(D) f_*(q_i)$ that is inverse to the image of $m_i$. This defines a homomorphism $S \to D$. It is the only homomorphism that extends $f$ because an invertible element has only one inverse. □

The Cohn localisation in the case where $P_i$ and $Q_i$ are free for all $i \in I$ is introduced in [**8**]. In this case, the homomorphism to the Cohn localisation makes certain matrices over the ring invertible. The classical case of localisation is when all these free modules are of rank 1. Then the localisation makes certain elements of a ring invertible.

PROPOSITION 4.6.13. *Let $\mathcal{C}$ be a bicategory with finitely many objects. Let $F\colon \mathcal{C} \to \mathfrak{Rings}_{\mathrm{fp}}$ be a morphism. The Cohn localisation of the lax covariance ring of $F$ at the set of homomorphisms $\{\psi_g : g \in \mathcal{C}\}$ in* (4.6.1) *is a strong covariance ring for $F$. In particular, the strong covariance ring exists.*

PROOF. The discussion above shows this if $\mathcal{C}$ is a category. The arguments work in essentially the same way if $\mathcal{C}$ is a bicategory. Still, a strong covariant representation is the same as a lax one for which the bimodule maps in (4.6.1) are invertible. The lax covariance ring exists by Exercise 4.6.9. And the Cohn localisation at the set of homomorphisms $\{\psi_g : g \in \mathcal{C}\}$ makes the bimodule maps in (4.6.1) invertible. □

REMARK 4.6.14. Let $R_1$ be an $R_0$-bimodule and let $D$ carry a lax covariant representation of the homomorphism $\mathbb{N} \to \mathfrak{Rings}$ defined by $R_0$ and $R_1$. Since $D$ is always finitely generated and projective as a $D$-module, the $D$-module homomorphism $R_1 \otimes_{R_0} D \to D$ can only be invertible if the $D$-module $R_1 \otimes_{R_0} D$ is finitely generated and projective as well. Since $D$ may be arbitrary, we would not expect this unless $R_1$ is finitely generated and projective as a right $R_0$-module. This is why we do not expect strong covariance rings to be well behaved (or even exist) unless $R_1$ is finitely generated and projective.

**4.6.3. Leavitt path algebras of finite graphs.** Let us consider some simple homomorphisms $(\mathbb{N},+) \to \mathfrak{Rings}$. Such a homomorphism is determined by a ring $R_0$ and an $R_0$-bimodule $R_1$. Let $R_0 := \bigoplus_{v\in V} \mathbb{K}$ for a finite set $V$ and a field $\mathbb{K}$. Then an $R_0$-bimodule is the same as a bimodule over $R_0 \otimes R_0 \cong \bigoplus_{v,w\in V^2} \mathbb{K}$. This is just a family of $\mathbb{K}$-vector spaces $(R_1)_{v,w}$ indexed by pairs $(v,w) \in V^2$. We may choose bases in these $\mathbb{K}$-vector spaces and combine the bases to a set $E$. Each $e \in E$ belongs to $(R_1)_{v,w}$ for some $v,w \in V$, and then we write $s(e) := v$, $r(e) := w$. Thus we describe a bimodule over $R_0$ through a set $E$ with two maps $s,r\colon E \rightrightarrows V$. This is the same as a directed graph $\Gamma$ with vertex set $V$, and $E$ is its set of directed edges. We assume $V$ to be finite in order for $R_0$ to be a *unital* ring. Later, we shall need $E$ to be finite in order for the $R_0$-bimodule $R_1$ to belong to $\mathfrak{Rings}_{\mathrm{fp}}$.

The $R_0$-bimodule $R_1$ gives $R_0$-bimodules $R_n := R_1^{\otimes_{R_0} n}$ for all $n \in \mathbb{N}$, and then $S := \bigoplus_{n\in\mathbb{N}} R_n$ becomes an $\mathbb{N}$-graded ring. This graded ring describes the homomorphism $\mathbb{N} \to \mathfrak{Rings}$ generated by $R_0$ and $R_1$. Here $R_n$ has the basis

$$\mathcal{P}^n(E) := \{(e_1,\dots,e_n) \in E^n : s(e_{i+1}) = r(e_i) \text{ for } i = 1,\dots,n-1\},$$

the set of paths of length $n$ in $\Gamma$. The $k$-algebra $S$ is also called the *path algebra* of $\Gamma$. Proposition 4.6.4 implies that $S$ is the lax covariance ring of the homomorphism $\mathbb{N} \to \mathfrak{Rings}$ described by $R_0$ and $R_1$.

Let $F,G\colon \mathbb{N} \rightrightarrows \mathfrak{Rings}$ be homomorphisms. Describe them by a ring $R_0$ with an $R_0$-bimodule $R_1$ and a ring $T_0$ with a $T_0$-bimodule $T_1$. A transformation $F \Rightarrow G$ is equivalent to an $R_0,T_0$-bimodule $M$ with a bimodule map $\tau_1\colon R_1 \otimes_{R_0} M \to M \otimes_{T_0} T_1$; this gives the maps

$$\tau_n\colon R_n \otimes_{R_0} M = R_1^{\otimes_{R_0} n} \otimes_{R_0} M \to M \otimes_{T_0} T_1^{\otimes_{T_0} n} \cong M \otimes_{T_0} T_n$$

for all $n \in \mathbb{N}$ by iteration. And $\tau_n$ is invertible for all $n \in \mathbb{N}$ once $\tau_1$ is invertible. Thus a lax covariant representation on $D$ is strong if and only if the bimodule map $R_1 \otimes_{R_0} S \to S$, $x \otimes y \mapsto x \cdot y$, induces an invertible map $R_1 \otimes_{R_0} D \xrightarrow{\simeq} R_0 \otimes_{R_0} D \cong D$; here we identify $S = \bigoplus_{n\in\mathbb{N}} R_n$ and use this to multiply $R_1$ and $S$.

PROPOSITION 4.6.15. *Let $\Gamma$ be a finite directed graph for which the map $s\colon E \to V$ is surjective. The Cohn localisation of the path algebra $S$ of $\Gamma$ at the map $R_1 \otimes_{R_0} S \to S$ above is the Leavitt path algebra of $\Gamma$.*

PROOF. We derive a presentation of the Cohn localisation of $S$. This gives the usual presentation of the Leavitt path algebra by generators and relations.

The path algebra $S$ of $\Gamma$ is the universal nonunital ring generated by an orthogonal family of idempotent elements $g_v$ for $v \in V$ and elements $g_e$ for $e \in E$, subject to the relations

$$g_v g_e = \delta_{v,s(e)} g_e, \qquad g_e g_v = \delta_{v,r(e)} g_e,$$

for all $e \in E$. In addition, $g_v g_w = \delta_{v,w} g_v$ for $v,w \in V$ expresses that the elements $g_v$ are orthogonal idempotents. We have assumed $V$ to be finite for technical reasons. Then $\sum_{v\in V} g_v$ is a unit element in $S$ because of the relations above. The bimodule $R_1 \otimes_{R_0} S$ is isomorphic to the direct sum $\sum_{e\in E} g_{r(e)} S$. Thus it is finitely generated and projective if and only if $E$ is finite. We have also assumed this. The Cohn localisation of $S$ occurs at the map $\sum_{e\in E} g_{r(e)} S \to S$ that multiplies on the left with the row vector $(g_e)_{e\in E}$. The localisation $S'$ has extra generators that correspond to the entries of the inverse of the induced map $\sum_{e\in E} g_{r(e)} S' \to S'$. The inverse map is pinned down by its value on $1 \in S'$, which is a column vector $(g_e^*)_{e\in E}$ with $g_e^* \in g_{r(e)} \cdot S'$, that is, $g_{r(e)} g_e^* = g_e^*$. In addition, the map of left multiplication by $(g_e^*)_{e\in E}$ is inverse to the map of left multiplication by $(g_e)_{e\in E}$. This means that $g_f^* g_e = \delta_{e,f} g_{r(e)}$ and that $\sum_{e\in E} g_e g_e^* = 1 = \sum_{v\in V} g_v$. Since $g_e \in g_{s(e)} S'$, the second relation is equivalent to $\sum_{e\in s^{-1}(v)} g_e g_e^* = g_v$ for all $v \in V$. In particular,

$g_e^* g_e = g_{r(e)}$, and this implies $g_v g_e^* = \delta_{v,r(e)} g_e^*$ for all $e \in E$, $v \in V$. The relations $\sum_{e \in s^{-1}(v)} g_e g_e^* = g_v$ imply $g_e^* g_v = \delta_{v,s(e)} g_e^*$ for all $e \in E$, $v \in V$. Thus our Cohn localisation is the universal ring generated by $g_v$ for $v \in V$ and $g_e, g_e^*$ for $e \in E$ subject to the following relations:

- $g_v g_w = \delta_{v,w} g_v$ for all $v, w \in V$;
- $g_v g_e = \delta_{v,s(e)} g_e$ and $g_e g_v = \delta_{v,r(e)} g_e$ for all $e \in E$;
- $g_f^* g_e = \delta_{e,f} g_{r(e)}$ for all $e, f \in E$;
- $\sum_{e \in s^{-1}(v)} g_e g_e^* = g_v$ for all $v \in V$.

And these imply $g_e^* g_v = \delta_{v,s(e)} g_e^*$ and $g_v g_e^* = \delta_{v,r(e)} g_e^*$ for all $e \in E$, $v \in V$.

This is the presentation of the Leavitt path algebra of the graph $\Gamma$ in [**2**, Definition 1.2.3], provided that $s\colon E \to V$ is surjective. □

The definition of the Leavitt path algebra through Cohn localisation is used first in [**4**, **5**].

If $s$ is not surjective, then there is $v \in V \setminus s(E)$. The relation $\sum_{e \in s^{-1}(v)} g_e g_e^* = g_v$ becomes $g_v = 0$. These relations of the strong covariance algebra are left out in the definition of a Leavitt path algebra in order to get a more interesting algebra.

Remark 4.6.16. If $V$ is finite and $E$ is infinite, then there are too few strong covariant representations. In particular, the obvious covariant representation of our homomorphism $\mathbb{N} \to \mathfrak{Rings}$ on the Leavitt path algebra of $\Gamma$ is *not* a strong transformation.

Remark 4.6.17. The results above generalise to the case where $V$ is infinite and the map $s\colon E \to V$ has finite, non-empty fibres. This requires the bicategory of self-induced nonunital rings and smooth bimodules in Section 4.6.5.

**4.6.4. Group actions by ring automorphisms.** How are the covariance rings above related to the covariance algebras for group actions on C*-algebras? There is, of course, a purely algebraic analogue of the crossed product. In this section, we turn an action of a group $G$ by automorphisms into a homomorphism $G \to \mathfrak{Rings}$. We identify the crossed product for the group action with the covariance ring of the homomorphism. We define twisted actions of $G$ by analysing a class of morphisms $G \to \mathfrak{Rings}$.

Let $G$ be a discrete group and let $R$ be a ring. A classical group action of $G$ on $R$ is a group homomorphism $\varrho\colon G \to \mathrm{Aut}(R)$, where $\mathrm{Aut}(R)$ denotes the group of ring isomorphisms $R \to R$. First we translate each ring automorphism $\varrho_g\colon R \to R$ into an $R,R$-bimodule $R_{\varrho_g}$ as in Section 4.3.7. Since the homomorphism in Lemma 4.3.17 involves opposites, we define $R_g := R_{\varrho_{g^{-1}}}$ for all $g \in G$. We may also describe this without inverses: the map $\varrho_g$ is an isomorphism from the bimodule $R_g$ above to the $R,R$-bimodule that is $R$ as a left $R$-module with the right $R$-module structure $r \cdot s := r \cdot \varrho_g(s)$. We shall, however, stick to the first description above to be definite.

To make the bimodules $R_g$ into a strictly unital homomorphism $G \to \mathfrak{Rings}$, we use the multiplication maps

$$\mu_{g,h}\colon R_g \otimes_R R_h = R_{\varrho_{g^{-1}}} \otimes_R R_{\varrho_{n^{-1}}} \xrightarrow{\simeq} R_{gh}, \qquad s \otimes r \mapsto \varrho_{g^{-1}}(s) \cdot r,$$

from the proof of Lemma 4.3.17.

Exercise 4.6.18. *Check that the data above is a strictly unital homomorphism of bicategories $\varrho_*\colon G \to \mathfrak{Rings}$. That is, the bimodule isomorphisms $\mu_{g,h}$ and $\lambda_x = \mathrm{id}_R$ satisfy the coherence conditions in Definition* 4.3.1.

The homomorphism $\varrho_*\colon G \to \mathfrak{Rings}$ corresponds to a $G$-graded ring $S$ by Proposition 4.4.3. The proof shows that $S = \bigoplus_{g \in G} R_g$ with the multiplication induced by the maps $\mu_{g,h}$. Inspection shows that $S$ is isomorphic to the crossed product $R \rtimes_\varrho G$: the isomorphism maps $r \in R_g$ to $\varrho_g(r) \cdot \delta_g = \delta_g \cdot r \in R \rtimes_\varrho G$. We may also reverse the reasoning above and start with the crossed product $R \rtimes_\varrho G$. This carries an obvious $G$-grading, which then defines a morphism $G \to \mathfrak{Rings}$. This gives the homomorphism above, up to natural isomorphism.

We may also turn the homomorphism $\varrho\colon G \to \mathrm{Aut}(R)$ into a homomorphism $G \to \mathfrak{Rings}$ using the product of homomorphisms defined in Proposition 4.7.10. Here we compose the homomorphism $\varrho\colon G^{\mathrm{op}} \to \mathrm{Aut}(R)^{\mathrm{op}}$ with the homomorphism from the opposite category of rings to $\mathfrak{Rings}$ in Lemma 4.3.17 and the isomorphism $G \xrightarrow{\cong} G^{\mathrm{op}}$, $g \mapsto g^{-1}$.

Proposition 4.3.9 says that all transformations between two homomorphisms $G \to \mathfrak{Rings}$ are strong. In particular, this applies to transformations $\mathrm{const}_D \Rightarrow \varrho_*$. Thus the strong and lax covariance rings for our homomorphism coincide. The lax covariance ring is identified in Proposition 4.6.7 with the graded ring $S$ that describes $\varrho_*$. We have seen that this is the crossed product $R \rtimes_\varrho G$. Thus the crossed product for a group action by automorphisms is both a lax and a strong covariance ring for the corresponding homomorphism $G \to \mathfrak{Rings}$.

It is easy to add a twist to the group homomorphisms. Consider a strictly unital morphism $A\colon G \to \mathfrak{Rings}$ with the extra property that the bimodules $A_g$ for $g \in G$ are all equal to the ring $R := A_1$ as left $R$-modules. The right $R$-module structure on $A_g = R$ must be of the form $a \cdot r := a\varrho_g(r)$ for all $a \in A_g = R$, $r \in R$, for a unique ring endomorphism $\varrho_g\colon R \to R$. And the multiplication map $\mu_{f,g}\colon A_f \otimes_R A_g \to A_{fg}$ for $f, g \in G$ must be of the form $\mu_{f,g}(a \otimes b) = a\varrho_f(b)u(f,g)$ for some $u(f,g) \in R$, namely, $u(f,g) = \mu_{f,g}(1 \otimes 1)$ (compare Lemma 4.3.17). We consider only strictly unital morphisms, that is, $\lambda_x = \mathrm{id}_R$. This forces $\varrho_1 = \mathrm{id}_R$. And (4.3.2) asserts that $u(f,g) = 1$ if $f = 1$ or $g = 1$. The associativity condition for a morphism becomes

$$a \cdot \varrho_f(b) \cdot u(f,g) \cdot \varrho_{fg}(c) \cdot u(fg,h) = a \cdot \varrho_f(b \cdot \varrho_g(c) \cdot u(g,h)) \cdot u(f,gh)$$

for all $f, g, h \in G$, $a, b, c \in R = A_f = A_g = A_h$. Here the factors $a$ and $b$ may be cancelled because $a = 1$ and $b = 1$ is possible. Taking $c = 1$ as well gives the condition

$$u(f,g) \cdot u(fg,h) = \varrho_f(u(g,h)) \cdot u(f,gh)$$

for $f, g, h \in G$. Taking $a = b = 1$ and $h = 1$ instead gives the condition

$$u(f,g) \cdot \varrho_{fg}(c) = \varrho_f\varrho_g(c) \cdot u(f,g)$$

for all $f, g \in G$, $c \in R$.

Exercise 4.6.19. *The morphism above is a homomorphism if and only if $\varrho_g \in \mathrm{Aut}(R)$ for all $g \in G$ and $u(f,g)$ is invertible for all $f, g \in G$.*

Let $R^\times$ be the group of invertible elements in $R$.

Definition 4.6.20. A *twisted action* of the group $G$ on the ring $R$ is a pair $(\varrho, u)$, consisting of maps $\varrho\colon G \to \mathrm{Aut}(R)$ and $u\colon G \times G \to R^\times$, such that:

- $u(f,g) \cdot u(fg,h) = \varrho_f(u(g,h)) \cdot u(f,gh)$ for all $f, g, h \in G$;
- $u(f,g) \cdot \varrho_{fg}(c) = \varrho_f\varrho_g(c) \cdot u(f,g)$ for all $f, g \in G$, $c \in R$.
- $\varrho_1 = \mathrm{id}_R$ and $u(1,g) = u(g,1) = 1$ for all $g \in G$.

These conventions differ a bit from those in Definition 2.3.2 because the intertwiners $u(f,g)$ above are 2-arrows $\varrho_{fg} \Rightarrow \varrho_f\varrho_g$. It would be possible to allow non-invertible $\varrho_g$ and $u(f,g)$ here. These would give a morphism $G \to \mathfrak{Rings}$ that is not a homomorphism.

**4.6.5. An excursion into nonunital rings.** If $\mathcal{C}$ is a category with infinitely many objects, then $\mathcal{C}$-graded rings are usually nonunital. So we need an appropriate bicategory of nonunital rings. This is also interesting to study analogies to the C*-correspondence bicategory in Chapter 5, because the latter has nonunital C*-algebras as objects. Before we describe a useful nonunital version of $\mathfrak{Rings}$, we discuss one that does not work.

Let $R$ be a ring without unit. Define a left $R$-module to be an Abelian group with a biadditive multiplication map $R \times M \to M$ that satisfies $(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$ for all $r_1, r_2 \in R$, $m \in M$, and define right modules and bimodules similarly. Let $R^+$ be $R \oplus \mathbb{Z}$ with a multiplication as in Section 1.1. This is a unital ring. And $R$-modules are exactly the same as "unital" $R^+$-modules, that is, $R^+$-modules that satisfy the usual condition $1 \cdot m = m$.

Example 4.6.21. Define a bicategory of nonunital rings by taking all nonunital bimodules over them, with bimodule homomorphisms as 2-arrows. This is indeed a bicategory. It even becomes a subbicategory of $\mathfrak{Rings}$ when we adjoin units to nonunital rings as above. It is useless, however, because we really want to restrict to *nondegenerate* bimodules. This is visible in Proposition 4.6.4. So the bicategory of nonunital rings with all bimodules has too many arrows. The concept of equivalence in this bicategory is unsatisfactory as well. For instance, the ring $\mathbb{M}_\infty(R) = \bigcup_{n \in \mathbb{N}} \mathbb{M}_n(R)$ of finite matrices over a unital ring $R$ is not equivalent to $R$ because $R^+$ and $\mathbb{M}_\infty(R)^+$ are not Morita equivalent. (Compare Exercise 5.9.13 for a similar issue when adjoining units in the C*-correspondence bicategory.)

Example 4.6.21 teaches us the lesson that we need to restrict to "nondegenerate" bimodules over nonunital rings if we want the resulting bicategory to be relevant for the study of covariant representations or for Morita theory. There are several ways to define nondegenerate modules over a nonunital ring. Our desire to make a bicategory of nonunital rings and nondegenerate bimodules suggests the following. The unit arrow on a ring $R$ should be the ring $R$ itself. (In the bicategory in Example 4.6.21, the unit arrow is the degenerate bimodule $R^+$.) This is a unit arrow for an $R,S$-bimodule $M$ if and only if the canonical maps $R \otimes_R M \to M$ and $M \otimes_S S \to M$ are isomorphisms. Then the nonunital rings themselves must satisfy $R \otimes_R R \cong R$ and $S \otimes_S S \cong S$ for the proposed unit arrows to be allowed as arrows. There are certainly nonunital rings for which this fails. For instance, it often happens that $R^2 \neq R$, that is, $R$ is not the closed linear span of products $x \cdot y$ for $x, y \in R$; this says that the map $R \otimes_R R \to R$ is not surjective.

Definition 4.6.22 ([**9**, **20**]). A ring $R$ is called *self-induced* if the multiplication map induces an isomorphism $R \otimes_R R \xrightarrow{\cong} R$.

Definition 4.6.23 ([**9**, **20**]). Let $R$ be a self-induced ring. A left $R$-module $M$ is called *smooth* if the multiplication map induces an isomorphism $R \otimes_R M \xrightarrow{\cong} M$. Smooth right modules and bimodules over self-induced rings are defined similarly.

Exercise 4.6.24. *There is a bicategory with self-induced rings as objects, smooth $S,R$-bimodules as arrows from $R$ to $S$, and bimodule homomorphisms as 2-arrows. The product of arrows is the bimodule tensor product $\otimes_R$.*

Niels Grønbæk studies Morita equivalence for self-induced Banach algebras in [**9**], using the projective Banach space tensor product instead of the purely algebraic tensor product. The definition in [**20**] is more general, covering algebras in an arbitrary monoidal category. This contains both Banach algebras and rings as special cases. Morita equivalence for self-induced rings is also mentioned in passing by Joseph Taylor in [**25**].

A $\mathcal{C}$-graded ring as defined above is certainly self-induced. More generally, if each $S_x$ is a self-induced ring and each $S_\gamma$ is a smooth $S_{r(\gamma)}, S_{s(\gamma)}$-bimodule, then the $\mathcal{C}$-graded ring $\bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ is self-induced. The definition of a $\mathcal{C}$-graded ring in Definition 4.4.1 asks for "local unit elements" $1_x \in S_x$. Then $\bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ has a local unit as in [**1**]. In this situation, an $S$-module is smooth if and only if the map $S \otimes M \to M$ is surjective; we have called such left modules nondegenerate above. In general, however, it is not enough if the map $S \otimes M \to M$ is surjective. This says that the canonical map $S \otimes_S M \to M$ is surjective. But we also need this map to be injective.

One of the main results about Morita equivalence of unital rings is Theorem 4.1.16, which characterises when a right module is part of a Morita equivalence. This theorem changes drastically when unital rings are replaced by a larger class of rings because a unital ring may be equivalent to a nonunital ring. There are two main reasons why I did not develop the theory in Section 4.1 for self-induced rings from the start. First, the proof of Theorem 4.1.9 gets more complicated, and this result is not stated in the literature in this generality. Hence a proof in full detail would be required. Secondly, I have not worked out the analogues of Theorem 4.1.16 in this setting, and a Morita theory without such a result seemed too shallow to me. In fact, there seem to be several interesting analogues of Theorem 4.1.16, depending on the class of nonunital rings that is allowed. This seems an interesting project, but it would have led me too far away from the main goals of this book.

Why are there several versions of Theorem 4.1.16? The category of smooth modules over a general self-induced ring need not be very well behaved. The only general fact that we know is that the inclusion of the smooth modules into all nonunital modules has a right adjoint, namely, the *smoothening* functor $R \otimes_R \sqcup$ (see [**20**, Theorem 3.6]). It is unclear, however, whether this functor is exact. And it is unclear whether the canonical map $R \otimes_R M \to M$ for an $R$-module $M$ is always injective. Hence we may need to restrict further to self-induced rings with these extra properties. Or we may want $R$ to be projective as a right or left $R$-module or we may want it to be a ring with local units (see [**1**]). There may, however, be rings that are self-induced and Morita equivalent to a unital ring but, say, do not have local units. Thus for each class of more or less well behaved nonunital rings, there should be a version of Theorem 4.1.16 that describes when a smooth $R$-module for a ring $R$ in that class is part of a Morita equivalence to another ring in the same class. And the relevant class of bimodules will depend on the class of nonunital rings that we allow, even if $R$ is a usual unital ring.

### 4.7. Defining objects of bicategories through universal properties

We have defined covariance algebras for group actions on C*-algebras by a universal property and generalised this to twisted group actions and actions of crossed modules, even locally compact ones. The lax and strong covariance rings of morphisms to the bicategory of $\mathfrak{Rings}$ are also defined by a universal property. So far, these universal properties took place in ordinary categories, namely, the category of C*-algebras and their morphisms (nondegenerate *-homomorphisms to the multiplier algebra) and the category of rings and ring homomorphisms. It follows from the usual Yoneda Lemma that such definitions work, that is, the universal property specifies an object of a category uniquely up to a unique isomorphism.

I already hinted that the concept of a limit in a category has a strong and a lax bicategorical analogue and that the covariance algebras of twisted group actions and the covariance rings of morphisms to $\mathfrak{Rings}$ are examples of this. We now develop this description of lax and strong covariance rings.

First, we show that an object $c$ of a bicategory $\mathcal{C}$ defines a homomorphism $\mathbb{Y}(c)\colon \mathcal{C} \to \mathfrak{Cat}$, mapping an object $x \in \mathcal{C}^0$ to the category $\mathcal{C}(c, x)$ of arrows $c \to x$ with the 2-arrows between them. This is the bicategorical analogue of the functor from a category to the category of sets that is represented by an object. So a representation of a homomorphism $F\colon \mathcal{C} \to \mathfrak{Cat}$ is defined as an equivalence $\mathbb{Y}(c) \simeq F$ in $\mathrm{Hom}(\mathcal{C}, \mathfrak{Cat})$ for some $c \in \mathcal{C}^0$.

Secondly, we want to prove that a representation of a homomorphism $F$ is "natural" and unique in some sense, and we want to understand how to build such a representation from a "universal element" of $F$. In usual category theory, all this follows from the Yoneda Lemma. So we need a bicategorical analogue of it. We show that the map $c \mapsto \mathbb{Y}(c)$ is part of a homomorphism $\mathcal{C}^{\mathrm{op}} \to \mathrm{Hom}(\mathcal{C}, \mathfrak{Cat})$. And for any homomorphism $F\colon \mathcal{C} \to \mathfrak{Cat}$, the category of strong transformations $\mathbb{Y}(c) \Rightarrow F$ and modifications between them is equivalent to the category $F(c)$. Thus a representation $\mathbb{Y}(c) \simeq F$ is induced by a suitable object of the category $F(c)$. This result is the bicategorical analogue of the Yoneda Lemma for ordinary categories.

Finally, we need interesting homomorphisms to which we may apply the Yoneda Lemma. For instance, to define the limit of a homomorphism $F\colon \mathcal{C} \to \mathcal{D}$, we need a homomorphism $\mathcal{D} \to \mathfrak{Cat}$ that maps an object $d \in \mathcal{D}$ to the category of cones $\mathrm{Cone}(D, F)$. This category is the category of arrows from $\mathrm{const}_d$ to $F$ in the bicategory $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$. We show that the map $d \mapsto \mathrm{const}_d$ extends to a homomorphism

$$\mathrm{const}\colon \mathcal{D} \to \mathrm{Hom}(\mathcal{C}, \mathcal{D}).$$

And we define a composition for (homo)morphisms. The product of the homomorphism const and the homomorphism $\mathrm{Hom}(\mathcal{C}, \mathcal{D}) \to \mathfrak{Cat}$ represented by the object $F$ of $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ is the desired homomorphism

$$\mathrm{Cone}(\sqcup, F)\colon \mathcal{D} \to \mathfrak{Cat}.$$

Working in the bicategory $\mathrm{Mor}(\mathcal{C}, \mathcal{D})$ instead gives the analogous homomorphism $\mathrm{Cone}_{\mathrm{lax}}(\sqcup, F)$. This theory then allow to formulate the universal properties of lax limits and limits and, dually, of lax colimits and colimits. And if one of these objects – say the limit – exists for two morphisms $F, G$, then the category $\mathrm{Hom}(F, G)$ of strong transformations $F \Rightarrow G$ is mapped to the category $\mathcal{D}(\lim F, \lim G)$ of arrows and 2-arrows between the limits.

The Yoneda Lemma for bicategories is stated without proof by Street in [**24**, (1.9)]. Despite its fundamental importance, proofs are only available in unpublished Master's Theses (see [**12**, **21**]). Therefore, it seems useful to include the details here. We state a more technical version of the Yoneda Lemma that also gives partial information about morphisms and transformations instead of homomorphisms and strong transformations. None of the results above are particularly difficult. But a homomorphism of bicategories is a complicated thing, and so proofs are complicated.

**4.7.1. The homomorphism represented by an object.** Fix $a \in \mathcal{C}$. We are going to define a homomorphism

$$\mathbb{Y}(a)\colon \mathcal{C} \to \mathfrak{Cat}.$$

If $x \in \mathcal{C}^0$, then we let $\mathbb{Y}(a)(x)$ be the category $\mathcal{C}(a, x)$ that has arrows $a \to x$ as objects and 2-arrows among these as arrows. If $f \in \mathcal{C}(x, y)$, then we let $\mathbb{Y}(a)(f) := f_*\colon \mathcal{C}(a, x) \to \mathcal{C}(a, y)$ be the functor that composes arrows $a \to x$ with $f$ and composes 2-arrows horizontally with the unit 2-arrow $1_f$. Let $f \in \mathcal{C}(y, z)$ and $g \in \mathcal{C}(x, y)$ be composable arrows in $\mathcal{C}$. Then $\mu_{f,g}^{\mathbb{Y}(a)}$ must be a natural isomorphism between the functors $f_* \circ g_*$ and $(f \circ g)_*$ from $\mathcal{C}(a, x)$ to $\mathcal{C}(a, z)$. These functors map $h \in \mathcal{C}(a, x)$ to $f \circ (g \circ h)$ and $(f \circ g) \circ h$, respectively. We let $\mu_{f,g}^{\mathbb{Y}(a)}(h)$ be the inverse associator $f \circ (g \circ h) \Rightarrow (f \circ g) \circ h$ in $\mathcal{C}$. Since associators are natural, this defines a natural isomorphism $f_* \circ g_* \Rightarrow (f \circ g)_*$. For the same reason, the map $(f,g) \mapsto \mu^{\mathbb{Y}(a)}_{f,g}$ is natural for 2-arrows $f_1 \Rightarrow f_2$ and $g_1 \Rightarrow g_2$, as needed for a homomorphism. In addition, we need a natural isomorphism $\lambda_x$ from the identity functor on $\mathcal{C}(a,x)$ to the functor $(1_x)_*\colon \mathcal{C}(a,x) \to \mathcal{C}(a,x)$, $h \mapsto 1_x \circ h$. As its value at $h \in \mathcal{C}(a,x)$, we take the inverse of the left uniter $1_x \circ h \Rightarrow h$.

Lemma 4.7.1. *The data above defines a homomorphism $\mathbb{Y}(a)\colon \mathcal{C} \to \mathfrak{Cat}$. This is called the* homomorphism represented by $a \in \mathcal{C}^0$.

Proof. We have already checked that $\mu^{\mathbb{Y}(a)}$ is natural. We must check that the diagrams in (4.3.1) and (4.3.2) commute. They assert that some natural transformations are equal. This is checked pointwise on an object of $\mathbb{Y}(a)(x) = \mathcal{C}(a,x)$ for some $x \in \mathcal{C}^0$. Thus (4.3.1) is about an equality of 2-arrows for four composable arrows $f,g,h,k$ in $\mathcal{C}$. It turns out to be equivalent to the associator pentagon (4.2.4) in the definition of a bicategory. The two diagrams in (4.3.2) say that the following 2-arrows for two composable arrows $f,g \in \mathcal{C}(a,x)$ are unit 2-arrows:

$$f \circ g \xRightarrow{1_f \bullet l_g^{-1}} f \circ (1_x \circ g) \xRightarrow{\text{ass}^{-1}} (f \circ 1_x) \circ g \xRightarrow{r_f \bullet 1_g} f \circ g,$$

$$f \circ g \xRightarrow{l_{f\circ g}^{-1}} 1_y \circ (f \circ g) \xRightarrow{\text{ass}^{-1}} (1_y \circ f) \circ g \xRightarrow{l_f \bullet 1_g} f \circ g.$$

The first 2-arrow is the unit because of the commuting diagram (4.2.3) in the definition of a bicategory. The second 2-arrow is the unit by one of the diagrams in (4.2.5); these commute in any bicategory. □

The homomorphisms $\mathcal{C} \to \mathfrak{Cat}$ form a bicategory $\mathrm{Hom}(\mathcal{C}, \mathfrak{Cat})$. Next we extend the map $a \mapsto \mathbb{Y}(a)$ to a homomorphism

$$\mathbb{Y}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Hom}(\mathcal{C}, \mathfrak{Cat}).$$

Let $a,b \in \mathcal{C}^0$ and $\varphi \in \mathcal{C}(a,b)$. We are going to define a strong transformation

$$\mathbb{Y}(\varphi)\colon \mathbb{Y}(b) \Rightarrow \mathbb{Y}(a).$$

If $x \in \mathcal{C}^0$, then we define the functor

$$\begin{gathered}\mathbb{Y}(\varphi)(x)\colon \mathcal{C}(b,x) \to \mathcal{C}(a,x),\\ f \mapsto f \circ \varphi,\\ (\alpha\colon f_1 \Rightarrow f_2) \mapsto (\alpha \bullet 1_\varphi\colon f_1 \circ \varphi \Rightarrow f_2 \circ \varphi).\end{gathered}$$

If $f \in \mathcal{C}(x,y)$, then $\mathbb{Y}(a)(f) \circ \mathbb{Y}(\varphi)(x)$ and $\mathbb{Y}(\varphi)(y) \circ \mathbb{Y}(b)(f)$ are the functors $\mathcal{C}(b,x) \to \mathcal{C}(a,y)$ that map $h \mapsto f \circ (h \circ \varphi)$ and $h \mapsto (f \circ h) \circ \varphi$, respectively. The inverse associators $f \circ (h \circ \varphi) \Rightarrow (f \circ h) \circ \varphi$ combine to a natural isomorphism

$$\mathbb{Y}(\varphi)(f)\colon \mathbb{Y}(a)(f) \circ \mathbb{Y}(\varphi)(x) \Rightarrow \mathbb{Y}(\varphi)(y) \circ \mathbb{Y}(b)(f).$$

It is checked as in the proof of Lemma 4.7.1 that this defines a strong transformation $\mathbb{Y}(\varphi)\colon \mathbb{Y}(b) \Rightarrow \mathbb{Y}(a)$.

Let $a,b \in \mathcal{C}^0$ and $\varphi_1, \varphi_2 \in \mathcal{C}(a,b)$. Then a 2-arrow $\alpha\colon \varphi_1 \Rightarrow \varphi_2$ gives a modification

$$\mathbb{Y}(\alpha)\colon \mathbb{Y}(\varphi_1) \Rrightarrow \mathbb{Y}(\varphi_2).$$

This is defined simply by letting $\mathbb{Y}(\alpha)(x)$ for $x \in \mathcal{C}^0$ be the natural transformation $f \mapsto 1_f \bullet \alpha$. The naturality of associators implies that this is indeed a modification. This construction is clearly multiplicative for the vertical product of 2-arrows and the vertical product of modifications in Lemma 4.5.1, and it maps the unit 2-arrow to the unit modification.

Let $a,b,c \in \mathcal{C}^0$ and $\varphi \in \mathcal{C}(a,b)$, $\psi \in \mathcal{C}(b,c)$. We define an invertible modification

$$\mu^{\mathbb{Y}}_{\psi,\varphi}\colon \mathbb{Y}(\varphi) \circ \mathbb{Y}(\psi) \Rrightarrow \mathbb{Y}(\psi \circ \varphi).$$

Here $\mathbb{Y}(\varphi) \circ \mathbb{Y}(\psi)$ is the product of (strong) transformations that is defined above Lemma 4.5.2. The value of $\mu^{\mathbb{Y}}_{\psi,\varphi}$ at $x \in \mathcal{C}^0$ must be a natural transformation $\mathbb{Y}(\varphi)(x) \circ \mathbb{Y}(\psi)(x) \Rightarrow \mathbb{Y}(\psi \circ \varphi)(x)$. Its value on $h \in \mathcal{C}(c,x)$ must be a 2-arrow $(h \circ \psi) \circ \varphi \Rightarrow h \circ (\psi \circ \varphi)$, and we take the associator once again. We claim that this defines an invertible modification. The coherence condition for a modification involves $f \in \mathcal{C}(x,y)$ and then says that certain natural transformations are equal. This is checked on an objects $h \in \mathcal{C}(c,x)$. Unravelling the condition, it becomes the associator pentagon (4.2.4) for the four composable arrows $f, h, \psi, \varphi$ once again.

Let $a \in \mathcal{C}^0$. We define an invertible modification

$$\lambda^{\mathbb{Y}}_a \colon 1_{\mathbb{Y}(a)} \Rrightarrow \mathbb{Y}(1_a)$$

from the unit transformation on the homomorphism $\mathbb{Y}(a)$ to the transformation $\mathbb{Y}(1_a) \colon \mathbb{Y}(a) \Rightarrow \mathbb{Y}(a)$. The value of $\lambda^{\mathbb{Y}}_a$ at $x \in \mathcal{C}^0$ must be a natural isomorphism from the identity functor on $\mathcal{C}(a,x)$ to the functor $\mathcal{C}(a,x) \to \mathcal{C}(a,x)$, $h \mapsto h \circ 1_a$. At $h \in \mathcal{C}(a,x)$, we simply take the inverse of the right uniter $r_h \colon h \circ 1_a \Rightarrow h$. These form a natural transformation $\lambda^{\mathbb{Y}}_a(x)$ for each $x \in \mathcal{C}^0$. Letting $x \in \mathcal{C}^0$ vary gives a modification because of the commuting diagrams in (4.2.3).

THEOREM 4.7.2. *The data above is a homomorphism* $\mathbb{Y} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Hom}(\mathcal{C}, \mathfrak{Cat})$.

PROOF. The modifications $\mu^{\mathbb{Y}}_{\psi,\varphi}$ are natural because of the naturality of associators. We must also verify that the diagrams in (4.3.1) and (4.3.2) commute. Again, these assert that certain natural transformations are equal. This is checked pointwise, giving an extra arrow. The diagram in (4.3.1) says that two 2-arrows $(f \circ \psi) \circ (\varphi \circ \zeta)$ to $f \circ ((\psi \circ \varphi) \circ \zeta)$ built out of associators are equal, and this is the associator pentagon (4.2.4). The two diagrams in (4.3.2) say that the following 2-arrows are units:

$$f \circ \psi \Rightarrow (f \circ 1_x) \circ \psi \Rightarrow f \circ (1_x \circ \psi) \Rightarrow f \circ \psi,$$
$$f \circ \psi \Rightarrow (f \circ \psi) \circ 1_a \Rightarrow f \circ (\psi \circ 1_a) \Rightarrow f \circ \psi.$$

And this follows from the commuting diagrams (4.2.3) and (4.2.5). □

Replacing $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$ gives an analogous Yoneda homomorphism

$$\mathbb{Y}^{\mathrm{op}} \colon \mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat}).$$

COROLLARY 4.7.3. *Let $x, y, z \in \mathcal{C}^0$ and let $f \in \mathcal{C}(x,y)$ be an equivalence, so that $x$ and $y$ are equivalent. Then the strong transformation $\mathbb{Y}(f)$ is an equivalence and the homomorphisms $\mathbb{Y}(x)$ and $\mathbb{Y}(y)$ are equivalent in the bicategory* $\mathrm{Hom}(\mathcal{C}, \mathfrak{Cat})$. *The functors*

$$\mathbb{Y}(f)_z \colon \mathcal{C}(x,z) \to \mathcal{C}(y,z), \qquad g \mapsto g \circ f, \quad \alpha \mapsto \alpha \bullet 1_f,$$

*for $z \in \mathcal{C}^0$ are equivalences of categories.*

PROOF. The first claim follows from Exercise 4.3.5. Using the inverse equivalence of $\mathbb{Y}(f)$, we show easily that the arrows $\mathbb{Y}(f)_z \colon \mathbb{Y}(x)_z \to \mathbb{Y}(y)_z$ are equivalences in $\mathfrak{Cat}$ for all $z \in \mathcal{C}^0$ (compare also Theorem 4.10.11). □

**4.7.2. Natural transformations out of represented functors.** Let $\mathcal{C}$ be a bicategory, $a \in \mathcal{C}^0$, and let $F \colon \mathcal{C} \to \mathfrak{Cat}$ be a morphism. Then transformations $\mathbb{Y}(a) \Rightarrow F$ and the modifications between them form a category $\mathrm{Mor}(\mathbb{Y}(a), F)$, namely, the categories of arrows and 2-arrows in the bicategory $\mathrm{Mor}(\mathcal{C}, \mathfrak{Cat})$. We are going to relate this category to the category $F(a)$. If $F$ is a *homo*morphism, then $F(a)$ turns out to be equivalent to the subcategory $\mathrm{Hom}(\mathbb{Y}(a), F)$ of *strong* transformations $\mathbb{Y}(a) \Rightarrow F$ and modifications between them. We first formulate what we are going to construct:

THEOREM 4.7.4. *Let $\mathcal{C}$ be a bicategory, $a \in \mathcal{C}^0$, and $F\colon \mathcal{C} \to \mathfrak{Cat}$ a morphism.*

(1) *There are functors*

$$\Phi\colon \operatorname{Mor}(\mathbb{Y}(a), F) \to F(a), \qquad \Psi\colon F(a) \to \operatorname{Mor}(\mathbb{Y}(a), F),$$

*and natural transformations*

$$\Lambda\colon 1_{F(a)} \Rightarrow \Phi \circ \Psi, \qquad \Gamma\colon \Psi \circ \Phi \Rightarrow 1_{\operatorname{Mor}(\mathbb{Y}(a),F)}.$$

(2) *If $F$ is a homomorphism, then the image of $\Psi$ consists only of strong transformations, and the natural transformations $\Lambda$ and $\Gamma$ above are invertible.*

(3) *If $\varphi\colon a_1 \to a_2$ is an arrow in $\mathcal{C}$, then there are natural transformations $m_1$ and $m_2$ that make the following squares of functors commute:*

$$\begin{array}{ccccc}
F(a_1) & \xrightarrow{\Psi} & \operatorname{Mor}(\mathbb{Y}(a_1), F) & \xrightarrow{\Phi} & F(a_1) \\
\Big\downarrow{\scriptstyle F(\varphi)} & \overset{m_1}{\Nearrow} & \Big\downarrow{\scriptstyle \mathbb{Y}(\varphi)^*} & \overset{m_2}{\Swarrow} & \Big\downarrow{\scriptstyle F(\varphi)} \\
F(a_2) & \xrightarrow[\Psi]{} & \operatorname{Mor}(\mathbb{Y}(a_2), F) & \xrightarrow[\Phi]{} & F(a_2)
\end{array}$$

*The natural transformation $m_1$ is invertible if $F$ is a homomorphism. The natural transformation $m_2$ becomes invertible on the subcategory $\operatorname{Hom}(\mathbb{Y}(a_1), F)$ of strong transformations.*

(4) *If $\sigma\colon F_1 \Rightarrow F_2$ is a transformation, then the following diagram of functors commutes up to a natural transformation $m_3$:*

$$\begin{array}{ccccc}
F_1(a_2) & \xrightarrow{\Psi} & \operatorname{Mor}(\mathbb{Y}(a_2), F_1) & \xrightarrow{\Phi} & F_1(a_2) \\
\Big\downarrow{\scriptstyle \sigma_{a_2}} & \overset{m_3}{\Nearrow} & \Big\downarrow{\scriptstyle \sigma_*} & = & \Big\downarrow{\scriptstyle \sigma_{a_2}} \\
F_2(a_2) & \xrightarrow[\Psi]{} & \operatorname{Mor}(\mathbb{Y}(a_2), F_2) & \xrightarrow[\Phi]{} & F_2(a_2)
\end{array}$$

*The natural transformation $m_3$ is invertible if $\sigma$ is strong.*

PROOF. We first define $\Phi$. Let $\tau\colon \mathbb{Y}(a) \Rightarrow F$ be a transformation. It contains a functor $\tau_a\colon \mathcal{C}(a, a) = \mathbb{Y}(a)(a) \to F(a)$. Let

$$\Phi(\tau) := \tau_a(1_a) \in F(a).$$

A modification $\mu\colon \tau_1 \Rrightarrow \tau_2$ contains a natural transformation $\mu_a\colon \tau_{1,a} \Rightarrow \tau_{2,a}$. Let

$$\Phi(\mu) := \mu_a(1_a)\colon \Phi(\tau_1) = \tau_{1,a}(1_a) \to \tau_{2,a}(1_a) = \Phi(\tau_2).$$

This is a functor $\operatorname{Mor}(\mathbb{Y}(a), F) \to F(a)$.

We define the functor $\Psi\colon F(a) \to \operatorname{Mor}(\mathbb{Y}(a), F)$. It consists of transformations $\Psi(\xi)\colon \mathbb{Y}(a) \Rightarrow F$ for $\xi \in F(a)$ and modifications $\Psi(\varphi)\colon \Psi(\xi_1) \Rrightarrow \Psi(\xi_2)$ for arrows $\varphi\colon \xi_1 \to \xi_2$ in $F(a)$ that are multiplicative for the product in $F(a)$ and the vertical product of modifications. Let $\xi \in F(a)$. The transformation $\Psi(\xi)$ consists of functors

$$\Psi(\xi)_x\colon \mathcal{C}(a, x) = \mathbb{Y}(a)(x) \to F(x)$$

for all $x \in \mathcal{C}^0$ and natural transformations

$$\Psi(\xi)_f\colon F(f) \circ \Psi(\xi)_x \Rightarrow \Psi(\xi)_y \circ \mathbb{Y}(a)(f)$$

for all $f \in \mathcal{C}(x, y)$, $x, y \in \mathcal{C}^0$. We define $\Psi(\xi)_x(h) := F(h)(\xi) \in F(x)$ for $h \in \mathcal{C}(a, x)$ and $\Psi(\xi)_x(\alpha) := F(\alpha)(\xi)\colon F(h_1)(\xi) \to F(h_2)(\xi)$ for a 2-arrow $\alpha\colon h_1 \Rightarrow h_2$ between $h_1, h_2 \in \mathcal{C}(a, x)$. This is a functor because $F$ is a morphism. If $f \in \mathcal{C}(x, y)$, then the composite functors $F(f) \circ \Psi(\xi)_x$ and $\Psi(\xi)_y \circ \mathbb{Y}(a)(f)$ from $\mathcal{C}(a, x)$ to $F(y)$ map $h \in \mathcal{C}(a, x)$ to $F(f)\bigl(F(h)(\xi)\bigr)$ and $F(f \circ h)(\xi)$, respectively. The datum $\mu_{f,h}\colon F(f) \circ F(h) \Rightarrow F(f \circ h)$ of the morphism $F$ provides the value of the natural transformation $\Psi(\xi)_f$ at $h$. The naturality of $\mu_{f,h}$ implies that this is a natural transformation and that it is natural for 2-arrows $f_1 \Rightarrow f_2$. To show that the functors $\Psi(\xi)_x$ and the natural transformations $\Psi(\xi)_f$ above form a transformation, we must check the two coherence diagrams (4.3.4) and (4.3.5) for a transformation. These assert equalities of natural transformations, which we check pointwise on an object of $\mathbb{Y}(a)(x) = \mathcal{C}(a,x)$. The first diagram (4.3.4) asserts that two arrows $F(f) \circ F(g) \circ F(h)(\xi) \to F((f \circ g) \circ h)(\xi)$ defined using only the data of a morphism are equal; this is first coherence diagram (4.3.1) for a morphism. The second diagram (4.3.2) asserts that two arrows $F(h)(\xi) \to F(1_x \circ h)(\xi)$ defined using only the data of a morphism are equal. Since $F(l_h)\colon F(1_x \circ h) \to F(h)$ is invertible, this also follows from the diagrams (4.3.1) and (4.3.2) for a morphism. Thus we have defined a transformation $\Psi(\xi)\colon \mathbb{Y}(a) \Rightarrow F$ for $\xi \in F(a)$. By construction, this transformation is strong if $F$ is a homomorphism.

Let $\xi_1, \xi_2 \in F(a)$ and let $\alpha\colon \xi_1 \to \xi_2$ be an arrow in $F(a)$. If $x \in \mathcal{C}^0$, $h \in \mathcal{C}(a,x)$, then $F(h)(\alpha)$ is an arrow $F(h)(\xi_1) \to F(h)(\xi_2)$ in $F(x)$. These arrows for $h \in \mathcal{C}(a,x)$ form a natural transformation $\Psi(\xi_1)_x \Rightarrow \Psi(\xi_2)_x$. These natural transformations for all $x \in \mathcal{C}^0$ form a modification

$$\Psi(\alpha)\colon \Psi(\xi_1) \Rrightarrow \Psi(\xi_2)$$

because each $\mu_{f,g}$ is a natural transformation, hence compatible with such arrows $\alpha$. The map $\alpha \mapsto \Psi(\alpha)$ is clearly multiplicative for the product in $F(a)$ and the vertical product of modifications. This finishes the construction of the functor $\Psi\colon F(a) \to \mathrm{Mor}(\mathbb{Y}(a), F)$. We have seen along the way that its image consists of strong transformations if $F$ is a homomorphism.

We define the natural transformation $\Lambda\colon 1_{F(a)} \to \Phi \circ \Psi$. Let $\xi \in F(a)$. Then $\Phi \circ \Psi(\xi) = F(1_a)(\xi)$. We let $\Lambda := \lambda_a^F\colon 1_{F^0(a)} \Rightarrow F(1_a)$ from the data of a morphism. This natural transformation is invertible if $F$ is a homomorphism.

We define the natural transformation $\Gamma\colon \Psi \circ \Phi \Rightarrow 1_{\mathrm{Mor}(\mathbb{Y}(a),F)}$. Let $\tau\colon \mathbb{Y}(a) \Rightarrow F$ be a transformation. The transformation $\Psi \circ \Phi(\tau)$ consists of functors

$$\Psi \circ \Phi(\tau)_x\colon \mathcal{C}(a,x) = \mathbb{Y}(a)(x) \to F(x)$$

and certain natural transformations $\Psi \circ \Phi(\tau)_f$. Here the functor $\Psi \circ \Phi(\tau)_x$ maps $h \in \mathcal{C}(a,x)$ to $F(h)(\tau_a(1_a))$. The natural transformation $\tau_h\colon F(h) \circ \tau_a \Rightarrow \tau_x \circ \mathbb{Y}(a)(h)$ in the transformation $\tau$ contains an arrow $F(h)(\tau_a(1_a)) \to \tau_x(h \circ 1_a)$. We let $\Gamma_{\tau,x,h}$ be its product with $\tau_x(r_h)\colon \tau_x(h \circ 1_a) \to \tau_x(h)$. These arrows for $h \in \mathcal{C}(a,x)$ combine to a natural transformation $\Gamma_{\tau,x}\colon \Psi \circ \Phi(\tau)_x \Rightarrow \tau_x$. And these natural transformations for $x \in \mathcal{C}^0$ form a modification $\Gamma_\tau\colon \Psi \circ \Phi(\tau) \Rrightarrow \tau$ because $\tau$ is a transformation. This modification is invertible if and only if each $\Gamma_{\tau,x}$ is invertible, if and only if each $\Gamma_{\tau,x,h}$ is invertible. And this follows if $\tau$ is a strong transformation.

Next we prove that $\Phi$ and $\Psi$ are natural for an arrow $\varphi\colon a_1 \to a_2$. We still write $F$ for the morphism $\mathcal{C} \to \mathfrak{Cat}$. The arrow $\varphi$ induces a functor $F(\varphi)\colon F(a_1) \to F(a_2)$ and a strong transformation $\mathbb{Y}(\varphi)\colon \mathbb{Y}(a_2) \Rightarrow \mathbb{Y}(a_1)$. Composition with $\mathbb{Y}(\varphi)$ is a functor $\mathbb{Y}(\varphi)^*\colon \mathrm{Mor}(\mathbb{Y}(a_1), F) \to \mathrm{Mor}(\mathbb{Y}(a_2), F)$. The functor $\mathbb{Y}(\varphi)^* \circ \Psi$ maps $\xi \in F(a_1)$ to the transformation $\Psi(\xi) \circ \mathbb{Y}(\varphi)\colon \mathbb{Y}(a_2) \Rightarrow F$. This consists of functors $T_x\colon \mathcal{C}(a_2,x) = \mathbb{Y}(a_2)(x) \to F(x)$ for $x \in \mathcal{C}^0$ and natural transformations $T_f\colon F(f) \circ T_x \Rightarrow T_y \circ \mathbb{Y}(a_2)(f)$ for $f \in \mathcal{C}(x,y)$, $x,y \in \mathcal{C}^0$. By construction, the functor $T_x$ maps $h \in \mathcal{C}(a_2,x)$ to $F(h \circ \varphi)(\xi)$ and $\alpha\colon h_1 \Rightarrow h_2$ to $F(\alpha \bullet 1_\varphi)_\xi$. And the natural transformation $T_f$ consists of the arrows

$$F(f) \circ F(h \circ \varphi)(\xi) \xrightarrow{\mu_{f,h\circ\varphi}(\xi)} F(f \circ (h \circ \varphi))(\xi) \xrightarrow{F(\mathrm{ass}^{-1})(\xi)} F((f \circ h) \circ \varphi)(\xi).$$

The functor $\Psi \circ F(\varphi)$ maps $\xi \in F(a_1)$ to the transformation $\Psi(F(\varphi)\xi)\colon \mathbb{Y}(a_2) \Rightarrow F$. This consists of functors $S_x\colon \mathcal{C}(a_2,x) = \mathbb{Y}(a_2)(x) \to F(x)$ for $x \in \mathcal{C}^0$ and natural transformations $S_f\colon F(f) \circ S_x \Rightarrow S_y \circ \mathbb{Y}(a_2)(f)$ for $f \in \mathcal{C}(x,y)$, $x,y \in \mathcal{C}^0$. By construction, the functor $S_x$ maps $h \in \mathcal{C}(a_2, x)$ to $F(h) \circ F(\varphi)(\xi)$ and $\alpha\colon h_1 \Rightarrow h_2$ to $F(\alpha)_{F(\varphi)\xi}$. And the natural transformation $S_f$ consists of the arrows

$$F(f) \circ F(h) \circ F(\varphi)(\xi) \xrightarrow{\mu_{f,h}(F(\varphi)\xi)} F(f \circ h) \circ F(\varphi)(\xi) \xrightarrow{\mu_{f\circ h,\varphi}(\xi)} F((f \circ h) \circ \varphi)(\xi).$$

As a consequence, the datum $\mu_{h,\varphi}$ for a morphism gives natural transformations $S_x \Rightarrow T_x$ for all $x \in \mathcal{C}^0$. These form a modification from the transformation $\Psi(F(\varphi)\xi)\colon \mathbb{Y}(a_2) \Rightarrow F$ to the transformation $\Psi(\xi) \circ \mathbb{Y}(\varphi)\colon \mathbb{Y}(a_2) \to F$ because $F$ is a morphism. These modifications are natural for arrows $\xi_1 \to \xi_2$ in $F(a_1)$. Thus they define a natural transformation that makes the first square in our naturality diagram commute. This natural transformation is invertible by construction if $F$ is a homomorphism.

Now we compare $F(\varphi) \circ \Phi$ and $\Phi \circ \mathbb{Y}(\varphi)^*$. These functors map a transformation $\tau\colon \mathbb{Y}(a_1) \Rightarrow F$ to $F(\varphi)\tau_{a_1}(1_{a_1})$ and $(\tau \circ \mathbb{Y}(\varphi))_{a_2}(1_{a_2}) = \tau_{a_2}(1_{a_2} \circ \varphi)$, respectively. The data of the transformation $\tau$ contains a natural transformation $\tau_\varphi\colon F(\varphi) \circ \tau_{a_1} \Rightarrow \tau_{a_2} \circ \mathbb{Y}(a_1)(\varphi)$. Its value at $1_{a_1} \in \mathbb{Y}(a_1)(a_1) = \mathcal{C}(a_1, a_1)$ is a natural arrow $F(\varphi)\circ\tau_{a_1}(1_{a_1}) \to \tau_{a_2}(\varphi\circ 1_{a_1})$. Since $\tau_{a_2}$ is functorial, it maps the uniter isomorphisms $\varphi \circ 1_{a_1} \cong \varphi \cong 1_{a_2} \circ \varphi$ in $\mathcal{C}$ to natural isomorphisms of functors. Combining these with $(\tau_\varphi)_{1_{a_1}}$ gives a natural transformation $F(\varphi) \circ \Phi \Rightarrow \Phi \circ \mathbb{Y}(\varphi)^*$. By construction, it is invertible on the subcategory of strong transformations.

Now we study functoriality for a transformation $\sigma\colon F_1 \Rightarrow F_2$. We write $a = a_1 = a_2$ to simplify notation. The functor $\sigma_* \circ \Psi$ maps $\xi \in F_1(a)$ to the transformation $\sigma \circ \Psi(\xi)$. It consists of functors $T_x\colon \mathcal{C}(a, x) = \mathbb{Y}(a)(x) \to F_2(x)$ and natural transformations $T_f\colon F_2(f) \circ T_x \Rightarrow T_y \circ \mathbb{Y}(a)(f)$ for $f \in \mathcal{C}(x, y)$, $x, y \in \mathcal{C}^0$. By definition, $T_x(h) = \sigma_x(F_1(h)(\xi))$ for all $h \in \mathcal{C}(a, h)$ and $T_f$ is the natural transformation

$$F_2(f)\sigma_x(F_1(h)\xi) \Rightarrow \sigma_y(F_1(f) \circ F_1(h)(\xi)) \Rightarrow \sigma_y(F_1(f \circ h)).$$

The functor $\Psi \circ \sigma_a$ maps $\xi \in F_1(a)$ to the transformation $\Psi(\sigma_a(\xi))$. It consists of functors $S_x\colon \mathcal{C}(a, x) = \mathbb{Y}(a)(x) \to F_2(x)$ and natural transformations $S_f\colon F_2(f) \circ S_x \Rightarrow S_y \circ \mathbb{Y}(a)(f)$ for $f \in \mathcal{C}(x, y)$, $x, y \in \mathcal{C}^0$. By definition, $S_x(h) = F_2(h)(\sigma_a(\xi))$ for all $h \in \mathcal{C}(a, h)$ and $S_f$ is the natural 2-arrow

$$\mu^{F_2}_{f,h} \bullet 1\colon F_2(f)F_2(h)\sigma_a(\xi) \Rightarrow F_2(f \circ h)\sigma_a(\xi).$$

The transformation $\sigma$ gives natural transformations $F_2(h) \circ \sigma_a \Rightarrow \sigma_x \circ F_1(h)$. Their values at all $h \in \mathcal{C}(a, x)$ combine to a natural transformation $S_x \Rightarrow T_x$. An easy computation shows that these natural transformations for all $x \in \mathcal{C}^0$ form a modification $\Psi \circ \sigma_a(\xi) \Rrightarrow (\sigma_* \circ \Psi)(\xi)$. These modifications are easily seen to be natural with respect to arrows in $F(a)$, so that they form a natural transformation $m_3$. By construction, it is invertible if and only if $\sigma$ is strong.

Both functors $\sigma_a \circ \Phi$ and $\Phi \circ \tau_*$ map a transformation $\tau\colon \mathbb{Y}(a) \Rightarrow F_1$ to $\sigma_a \circ \tau_a(1_a) = (\sigma \circ \tau)_a(1_a)$, and they also agree on modifications. So the fourth naturality square commutes exactly. □

Corollary 4.7.5. *Let $\mathcal{C}$ be a bicategory, let $a \in \mathcal{C}^0$, and let $F\colon \mathcal{C} \to \mathfrak{Cat}$ be a homomorphism. The functors and natural transformations $\Phi$, $\Psi$, $\Lambda$ and $\Gamma$ in Theorem 4.7.4 restrict to an equivalence of categories* $\mathrm{Hom}(\mathbb{Y}(a), F) \simeq F(a)$ *that is natural in $a$ and $F$. Here* Hom *denotes the category of strong transformations and modifications.*

Corollary 4.7.6. *If $a, b \in \mathcal{C}^0$, then the Yoneda functor*

$$\mathbb{Y}_{a,b}\colon \mathcal{C}(a, b) \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat})\big(\mathbb{Y}(a), \mathbb{Y}(b)\big)$$

*is an equivalence of categories.*

Proof. Take $F = \mathbb{Y}(b)$ in Corollary 4.7.5. □

We may now describe representations of functors through analogues of the universal elements in usual category theory (see [**23**, Section 2.3]).

DEFINITION 4.7.7. Let $\mathcal{C}$ be a bicategory and $F\colon \mathcal{C} \to \mathfrak{Cat}$ a homomorphism. A *representation* of $F$ is an equivalence $F \cong \mathbb{Y}(a)$ in the bicategory $\mathrm{Hom}(\mathcal{C}, \mathfrak{Cat})$. We call $F$ *representable* if there is such a representation.

THEOREM 4.7.8. *Let $\mathcal{C}$ be a bicategory. A homomorphism $F\colon \mathcal{C} \to \mathfrak{Cat}$ is representable if and only if there are $c \in \mathcal{C}^0$ and $\xi \in F(c)$ such that the transformation $\Psi(\xi)\colon \mathbb{Y}(c) \Rightarrow F$ is an equivalence. Equivalently, for each object $d \in \mathcal{C}^0$, the following functor is an equivalence of categories:*

$$\begin{aligned}\Psi(\xi)_d\colon \mathcal{C}(a,d) &\to F(d),\\ f &\mapsto F(f)(\xi),\\ (\alpha\colon f_1 \Rightarrow f_2) &\mapsto (F(\alpha)_\xi\colon F(f_1)(\xi) \to F(f_2)(\xi)),\end{aligned}$$

PROOF. Let $c \in \mathcal{C}^0$ and $\xi \in F(c)^0$. The transformation $\Psi(\xi)\colon \mathbb{Y}(c) \Rightarrow F$ is strong by Theorem 4.7.4 because $F$ is a homomorphism. By Theorem 4.10.11, it is an equivalence if and only if all the functors $\Psi(\xi)_d$ for $d \in \mathcal{C}^0$ are equivalences. If this happens, then the inverse of the equivalence $\Psi(\xi)\colon \mathbb{Y}(c) \simeq F$ is a representation of the homomorphism $F$. Conversely, if $F$ is representable, then there are $c \in \mathcal{C}^0$ and an equivalence $\tau\colon \mathbb{Y}(c) \Rightarrow F$. Let $\xi := \Phi(\tau) \in F(c)^0$. The transformation $\tau$ is strong by Theorem 4.10.11. By Corollary 4.7.5, $\tau$ is isomorphic to $\Psi \circ \Phi(\tau) = \Psi(\xi)$ in the category $\mathrm{Hom}(\mathbb{Y}(c), F)$. Therefore, the transformation $\Psi(\xi)$ is an equivalence. □

DEFINITION 4.7.9. If $c \in \mathcal{C}^0$ and $\xi \in F(c)^0$ are such that $\Psi(\xi)\colon \mathbb{Y}(c) \Rightarrow F$ is an equivalence as in Theorem 4.7.8, then $\xi$ is called a *universal object of $F$*.

Our next goal is to define the universal cone and the universal lax cone over a morphism $F\colon \mathcal{C} \to \mathcal{D}$. These then characterise the lax limit and the limit of the morphism $F$. To define them, we need to embed the construction of the categories $\mathrm{Cone}(d, F)$ and $\mathrm{Cone}_{\mathrm{lax}}(d, F)$ for $d \in \mathcal{D}^0$ in Definition 4.6.3 into a homomorphism $\mathcal{D} \to \mathfrak{Cat}$. This uses two ingredients. First, we are going to define how to compose morphisms in Section 4.7.3. Secondly, we are going to embed the map const in Definition 4.6.1 into a homomorphism

$$\mathrm{const}\colon \mathcal{D} \to \mathrm{Hom}(\mathcal{C}, \mathcal{D}).$$

**4.7.3. The composition of morphisms.** We are going to compose morphisms. This gives a category of bicategories. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be bicategories. Let $G\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{D} \to \mathcal{E}$ be morphisms. Its composite is a morphism $F * G\colon \mathcal{C} \to \mathcal{E}$. The definition on objects and arrows is the obvious one, namely, $(F*G)^0(x) := F^0(G^0(x))$ for all $x \in \mathcal{C}^0$ and $(F*G)(f) := F(G(f))$ for any arrow $f$ in $\mathcal{C}$. The 2-arrows $\mu_{g,f}$ and $\lambda_x$ are also defined in the simplest possibe way:

$$G(F(g)) \circ G(F(f)) \xRightarrow{\mu^G_{F(g),F(f)}} G(F(g) \circ F(f)) \xRightarrow{G(\mu^F_{g,f})} G(F(g \circ f)),$$

$$1_{G^0(F^0(x))} \xRightarrow{\lambda^G_{F^0(x)}} G(1_{F^0(x)}) \xRightarrow{G(\lambda^F_x)} G(F(1_x)).$$

These 2-arrows are clearly natural.

The *unit homomorphism* $1_{\mathcal{C}}$ on a bicategory $\mathcal{C}$ is defined by taking $F^0$ and $F$ to be identity maps and $\mu_{g,f}$ and $\lambda_x$ to be unit 2-arrows.

PROPOSITION 4.7.10 ([**6**, Section 4.3]). *The data above defines a morphism $F * G\colon \mathcal{C} \to \mathcal{E}$. The product $F * G$ is strong, strict, or strictly unital if both $F$ and $G$ have the corresponding property. The product above is strictly associative and strictly unital, that is, $(F * G) * H = F * (G * H)$ for three composable morphisms $F, G, H$,* *and* $1_{\mathcal{D}} * F = F = F * 1_{\mathcal{C}}$ *for any morphism* $F\colon \mathcal{C} \to \mathcal{D}$*. Thus there is a category with bicategories as objects and morphisms as arrows, and it has subcategories with homomorphisms, strictly unital morphisms, or strictly unital homomorphisms as arrows.*

Proof. To prove that $F * G$ is a morphism of bicategories, we must show that the diagrams in Figure 3 and Figure 4 commute. The two triangles on the left and the two triangles on the right in Figure 3 commute by definition of $F * G$. The two quadrangles commute because the 2-arrows $\mu^F$ are natural. The two hexagons in the middle commute because $F$ and $G$ are morphisms. Therefore, the outer hexagon in Figure 3 commutes. This is (4.3.1). The two diagrams in Figure 4 differ only by exchanging left and right, and the proof that they commute is essentially the same. The two triangles commute by the definition of $\mu^{F*G}$ and $\lambda^{F*G}$. The left quadrangle commutes because $F$ is a morphism and the top quadrangle commutes because $G$ is a morphism. The bottom right quadrangle commutes because $\mu^F$ is natural. Thus the outer rectangle commutes. This is another coherence condition in Definition 4.3.1. The three commuting diagrams in Figure 3 and Figure 4 show that $F * G$ is a morphism. It is elementary to check that the product of morphisms is associative and unital. And it is clear from the definitions that $F * G$ inherits the properties of being strong, strict, strongly unital, or strictly unital from $F$ and $G$. □

Exercise 4.7.11. *A morphism* $F\colon \mathcal{C} \to \mathcal{D}$ *between two bicategories is invertible in the category of bicategories and morphisms defined above if and only if it is an isomorphism in the naive sense, that is,* $F$ *is a homomorphism and it is bijective on objects, arrows and 2-arrows.*

**4.7.4. Universal cones and limits.** We are going to embed the construction of constant homomorphisms in Definition 4.6.1 into a strict homomorphism

$$\mathrm{const}\colon \mathcal{D} \to \mathrm{Hom}(\mathcal{C}, \mathcal{D}). \tag{4.7.1}$$

It maps $x \mapsto \mathrm{const}_x$ on objects. The "constant" transformation

$$\mathrm{const}_f\colon \mathrm{const}_x \Rightarrow \mathrm{const}_y$$

for an arrow $f \in \mathcal{D}(x, y)$ consists of the arrows $f\colon x = \mathrm{const}_x(c) \to \mathrm{const}_y(c) = y$ at all objects $c \in \mathcal{C}^0$ and the uniters $f \circ 1_x \Rightarrow f \Rightarrow 1_y \circ f$ for all arrows in $\mathcal{C}$. It is easy to check that this is a strong transformation. An arrow $f_1 \Rightarrow f_2$ in $\mathcal{D}$ induces a modification $\mathrm{const}_\alpha\colon \mathrm{const}_{f_1} \Rrightarrow \mathrm{const}_{f_2}$, which consists of the 2-arrows $\alpha\colon f_1 \Rightarrow f_2$ at all $c \in \mathcal{C}^0$. The map $f \mapsto \mathrm{const}_f$ above is strictly unital. We let $\mu_{f,g}$ and $\lambda_x$ in the definition of the morphism const be the unit 2-arrows.

Lemma 4.7.12. *The data above defines a homomorphism*

$$\mathrm{const}\colon \mathcal{D} \to \mathrm{Hom}(\mathcal{C}, \mathcal{D}).$$

Proof. This is elementary to check. □

The bicategory $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ is contained in the bicategory $\mathrm{Mor}(\mathcal{C}, \mathcal{D})$. In between, there is a $\mathrm{Mor}(\mathcal{C}, \mathcal{D})_{\mathrm{strong}}$ that has morphisms as objects, strong transformations as arrows, and modifications as 2-arrows. The given morphism $F$ is an object of the bicategories $\mathrm{Mor}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Mor}(\mathcal{C}, \mathcal{D})_{\mathrm{strong}}$ and, as such, represents homomorphisms

$$\mathbb{Y}(F)_{\mathrm{strong}}\colon \mathrm{Mor}(\mathcal{C}, \mathcal{D})_{\mathrm{strong}} \to \mathfrak{Cat}, \qquad \mathbb{Y}(F)\colon \mathrm{Mor}(\mathcal{C}, \mathcal{D}) \to \mathfrak{Cat}.$$

Finally, we define $\mathrm{Cone}(\llcorner\lrcorner, F) := \mathbb{Y}(F)_{\mathrm{strong}} * \mathrm{const}$ and $\mathrm{Cone}_{\mathrm{lax}}(\llcorner\lrcorner, F) := \mathbb{Y}(F) * \mathrm{const}$; these are homomorphisms $\mathcal{D} \to \mathfrak{Cat}$. By construction, $\mathrm{Cone}(\llcorner\lrcorner, F)(d)$ for $d \in \mathcal{D}^0$ is the category $\mathrm{Cone}(d, F)$ of cones over $F$ with summit $d$ and $\mathrm{Cone}_{\mathrm{lax}}(\llcorner\lrcorner, F)(d)$ is the category $\mathrm{Cone}_{\mathrm{lax}}(d, F)$ of lax cones over $F$ with summit $d$.

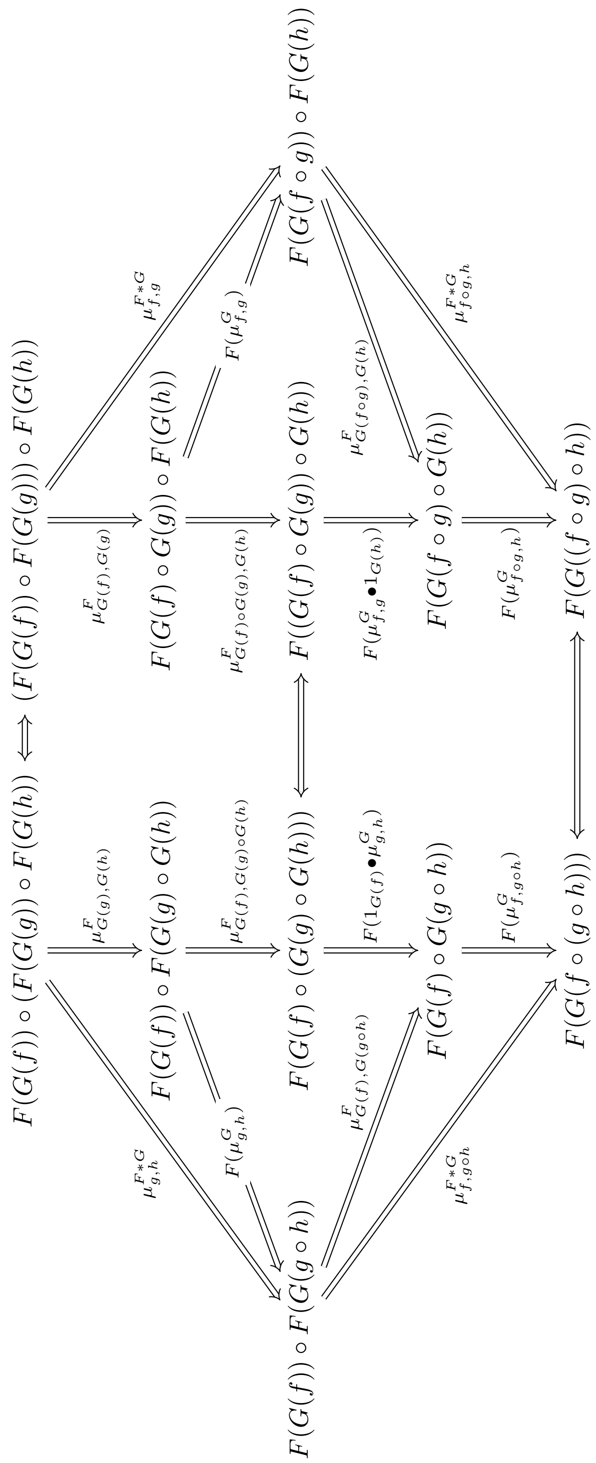



FIGURE 3. Proof that $F * G$ is a morphism, part I. The double-headed arrows are associators.

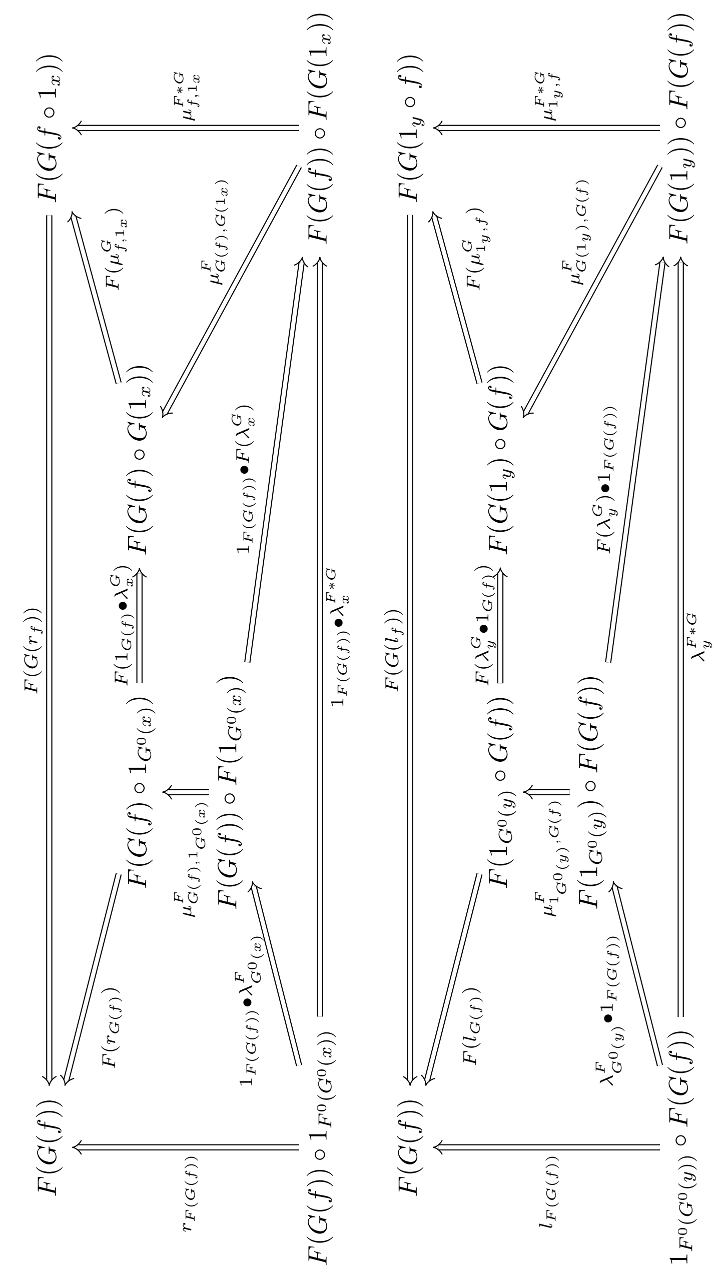



FIGURE 4. Proof that $F * G$ is a morphism, part II.

DEFINITION 4.7.13. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism. A *universal cone* over $F$ is a universal object of the homomorphism $\mathrm{Cone}_{\mathrm{lax}}(\sqcup, F)\colon \mathcal{D} \to \mathfrak{Cat}$. A *universal lax cone* over $F$ is a universal object of the homomorphism $\mathrm{Cone}(\sqcup, F)\colon \mathcal{D} \to \mathfrak{Cat}$. The object of $\mathcal{D}$ that represents the (lax) cone homomorphism is called a (lax) *limit* of $F$.

By definition, the universal cone is a pair $\lim F, \xi$, where $\lim F \in \mathcal{D}^0$ – the limit of $F$ – is an object and $\xi$ is a cone over $F$ with summit $\lim F$ – that is, a strong transformation $\xi\colon \mathrm{const}_{\lim F} \Rightarrow F$ – with the property that for each $d \in \mathcal{D}^0$, the functor $\mathcal{D}(d, \lim F) \to \mathrm{Cone}(d, F)$ induced by $\xi$ is an equivalence of categories (see Theorem 4.7.8); this functor is part of the homomorphism $\mathrm{Cone}(\llcorner, F)$. The definition for the universal lax cone and the lax limit is the similar, with lax cones and transformations instead instead of cones and strong transformations.

EXAMPLE 4.7.14. Let $\mathcal{C}$ be the bicategory associated to a discrete group $G$ or a discrete crossed module $\partial\colon H \to G$, $c\colon G \to \mathrm{Aut}(H)$. Let $\mathcal{D} = \mathcal{C}^*(2)$. We have identified a strictly unital homomorphism $F\colon \mathcal{C} \to \mathcal{C}^*(2)$ with a twisted action of $\mathcal{C}$ on a C*-algebra $A$ (see Example 4.3.2 and Exercise 4.3.3). We are interested in cones *under* such homomorphisms because the passage from homomorphisms to bimodules is contravariant. In this case, all lax cones are cones by Proposition 4.3.9. A (lax) cone under $F$ with nadir $D$ is a transformation $F \Rightarrow \mathrm{const}_D$ for a C*-algebra $D$. The constant homomorphism $\mathrm{const}_D$ is the homomorphism that belongs to the trivial $\mathcal{C}$-action on $D$. So cones under $F$ with nadir $D$ are the same as covariant representations of the twisted action underlying $F$ on $D$. By definition, such cones are in bijection with morphisms $A \rtimes \mathcal{C} \to D$. Given two such morphisms, a modification between the corresponding transformations is the same as a unitary multiplier of $D$ that intertwines the representations of $A$ and of the group $G$ underlying $\mathcal{C}$. This is equivalent to intertwining the two morphisms $A \rtimes \mathcal{C} \to D$. So the category $\mathrm{Cone}(F, \mathrm{const}_D)$ is isomorphic – not just equivalent – to the category $\mathcal{C}^*(2)(A \rtimes \mathcal{C}, D)$. This means that the crossed product $A \rtimes \mathcal{C}$ is a colimit of the homomorphism $F$.

PROPOSITION 4.7.15. *Let $\mathcal{C}$ be a bicategory with finitely many objects and let $F\colon \mathcal{C} \to \mathfrak{Rings}$ be a morphism. A lax covariance ring of $F$ is also a lax limit of this morphism, and a covariance ring of $F$ is also a limit.*

PROOF. We prove the result for covariance rings. The argument in the lax case is exactly the same, just omitting the adjective “strong” everywhere. Let $S$ be a covariance ring of $F$. Then the identity map on $S$ corresponds to a strong transformation $\xi\colon \mathrm{const}_S \Rightarrow F$. By assumption, if $D$ is any ring and $\tau\colon \mathrm{const}_D \Rightarrow F$ is a covariant representation of $F$ on $D$, then there is a unique ring homomorphism $f\colon S \to D$ so that $\tau = f_*(\xi)$. Recall that covariant representations are strong transformations with the extra property that $\tau_x = p_x \cdot D \subseteq D$ with idempotent elements $p_x \in D$ for $x \in \mathcal{C}^0$ that satisfy $\sum_{x\in\mathcal{C}^0} p_x = 1$. Now let $D_0$ be a ring and let $M$ be any $S, D_0$-bimodule. Let $D := \mathrm{End}(D_0)$. The left $S$-module structure on $M$ is the same as a unital ring homomorphism $f\colon S \to D$. And this is the same as a covariant representation of $S$ on $D$. The latter gives idempotent elements $p_x \in D$ and a strong transformation $\tau\colon \mathrm{const}_D \Rightarrow F$ such that the right $D$-modules underlying the bimodules $\tau_x$ for $x \in \mathcal{C}^0$ are $p_x \cdot D$. We view each $p_x$ as an idempotent map $M \to M$ and define $\tau'_x := p_x \cdot M \subseteq M$. These are right $D_0$-modules with $\sum_{x\in\mathcal{C}^0} \tau'_x = M$. If $x, y \in \mathcal{C}^0$, $g \in \mathcal{C}(x, y)$, then $\tau_g$ is an $F(y), F(x)$-bimodule and $\tau_f$ is a right $D$-module isomorphism

$$\tau_g\colon F(g) \otimes_{F(x)} p_x \cdot D \cong p_y \cdot D.$$

By Lemma 4.1.8, this isomorphism is equivalent to a group homomorphism

$$F(g) \to \mathrm{Hom}_D(p_x \cdot D, p_y \cdot D) \cong p_x \cdot D \cdot p_y \cong \mathrm{Hom}_{D_0}(p_x \cdot M, p_y \cdot M).$$

And another application of Lemma 4.1.8 turns the latter into a right $D_0$-module isomorphism

$$\tau'_g\colon F(g) \otimes_{F(x)} p_x \cdot M \cong p_y \cdot M.$$

Using the naturality of the adjoint associativity isomorphisms above, it follows that the maps $(\tau_g)$ form a strong transformation if and only if the maps $(\tau'_g)$ form a strong transformation. Therefore, the universal property of the covariance ring also describes $S, D_0$-bimodules: these are "equivalent" to strong transformations $\tau\colon \mathrm{const}_D \Rightarrow F$. Here "equivalent" means as usual that the appropriate categories are equivalent. We do not get an isomorphism of categories because direct sums are only unique up to canonical isomorphism and we are going back and forth between $M$ and the family $(M_x)_{x\in\mathcal{C}^0}$ with $M = \bigoplus_{x\in\mathcal{C}^0} M_x$. □

A ring may be a limit of a homomorphism $F\colon \mathcal{C} \to \mathfrak{Rings}$ without being a covariance ring. The most obvious reason for this is that if $R_1$ is a limit of $F$ and $R_1$ is Morita equivalent to $R_2$, then $R_2$ is a limit as well because then the homomorphisms represented by $R_1$ and $R_2$ are equivalent by Corollary 4.7.3. In contrast, the covariance ring is unique up to isomorphism, not just up to equivalence.

The following example clarifies in which sense limits of functors between categories are a special case of limits between homomorphisms of bicategories.

Example 4.7.16. Let $\mathcal{C}$ be a bicategory and let $\mathcal{D}$ be a category, viewed as a bicategory with only identity 2-arrows. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism. Then $F$ is automatically strict. And if $a\colon f \Rightarrow g$ is a 2-arrow in $\mathcal{C}$, then $F(a)$ is an identity 2-arrow, forcing $F(f) = F(g)$. So $F$ factors through a functor $F'\colon \mathcal{C}' \to \mathcal{D}$, where $\mathcal{C}'$ is the category that has the same objects as $\mathcal{C}$ and where two arrows are identified if there is a 2-arrow between them. (This category is not the same as the one in Exercise 4.2.7). A transformation between two morphisms $\mathcal{C} \rightrightarrows \mathcal{D}$ is automatically strong, and it is the same as a natural transformation between the corresponding functors $\mathcal{C}' \to \mathcal{D}$. And any modification is an identity because there are no non-identity 2-arrows in $\mathcal{D}$. As a result, a (lax) cones over $F$ is the same as a cone over $F'$ in the usual sense, and the category of cones over $F$ has only identity arrows. As a result, the universal property of the bicategory-theoretic (lax) limit of $F$ is the same as the universal property of the category-theoretic limit of the functor $F'$.

## 4.8. The Coherence Theorem for bicategories

When we introduced the weakening scheme in Section 2.4, we proposed to replace equalities between arrows in the definition of a classical concept by 2-arrows and impose coherence conditions whenever there are two ways to prove an equality between two arrows in the classical setting. We always wrote down only a few coherence conditions, however, and claimed that these implied all others. We now have the tools to prove these claims.

First we slightly modify the weakening scheme to take into account 2-arrows already present in a classical concept: namely, the 2-arrows that the weakening scheme adds to our data are required to be natural for 2-arrows. This naturality is empty for the twisted actions of groups studied in Section 2.4. In Section 2.7 on twisted actions of crossed modules, it was deduced from the weakening scheme, by treating an invertible 2-arrow $h\colon g \Rightarrow \partial(h)\cdot g$ in the 2-group associated to a crossed module as a way to prove the classical equality $g = \partial(h)g$. Once there are non-invertible 2-arrows, this seems less attractive, and so we simply make naturality a requirement for the weakening scheme.

The concept of a bicategory is a weakening of the concept of a 2-category. The associators and uniters of a bicategory replace the strict associativity and unitality conditions for a 2-category. We have just incorporated the naturality requirements for these 2-arrows in Definition 4.2.1 into the weakening scheme. The other conditions in Definition 4.2.1 come from two ways of proving the identities $(f_1 \circ 1) \circ f_2 = f_1 \cdot f_2$ and $((f_1 \circ f_2) \circ f_3) \circ f_4 = f_1 \circ (f_2 \circ (f_3 \circ f_4))$ in a 2-category. According to the weakening scheme, we should have added many more coherence conditions to the definiton of a bicategory. MacLane's Coherence Theorem for bicategories says that all these coherence conditions already follow from the two that are required in Definition 4.2.1. For instance, Lemma 4.2.4 proves this for the two coherence conditions in (4.2.5), which come from the two ways of proving $(f_1 \circ f_2) \circ 1 = f_1 \cdot f_2$ and $1 \circ (f_1 \circ f_2) = f_1 \cdot f_2$.

The following proof of the Coherence Theorem is based on a sketch by Leinster (see [**18**]). He treats only one "typical" example of a coherence condition and claims that the same argument works in general. We add more details to this argument. The key point in the proof is the Yoneda Embedding in Corollary 4.7.6. Roughly speaking, a homomorphism between two bicategories intertwines the associators and uniters in the two bicategories. If the homomorphism is faithful on 2-arrows, then the validity of a coherence condition in the target bicategory implies that it is also valid in the domain bicategory. The Yoneda Embedding is faithful on 2-arrows, and its target is a 2-category, where all coherence conditions become simply identities. Therefore, the existence of the Yoneda embedding implies that all coherence conditions that are produced by the weakening scheme hold in any bicategory. The proof that the Yoneda embedding exists also uses the coherence conditions in (4.2.5), in addition to the axioms of a bicategory. We proved these two consequences of the Coherence Theorem separately, because they are needed for our proof of the general Coherence Theorem.

To make the Coherence Theorem more precise, we need *non-associative words* in a bicategory $\mathcal{C}$. These are formal expressions such as $h(gf)$ for arrows $f \in \mathcal{C}(x,y)$, $g \in \mathcal{C}(y,z)$, $h \in \mathcal{C}(z,w)$ or $(h(I_z g))f$; here the symbol $I_z$ stands for a formal unit arrow on the object $z$. Parentheses are put to fix the order in which products in $\mathcal{C}$ are to be performed. The whole word must be composable, that is, the ranges and sources of consecutive letters must match; here $r(I_z) = s(I_z) = z$. A non-associative word in $\mathcal{C}$ *evaluates* to an arrow in $\mathcal{C}$ by replacing each $I_z$ by the unit arrow $1_z$ and then multiplying all the letters as specified by the parentheses. We denote the evaluation of a word $w$ by $\mathrm{ev}(w)$. Two non-associative words are only identified when they are truly equal. For instance, $f(gh)$ and $(fg)h$ are different non-associative words, and so are $1_z$, the unit arrow on $z$ and $I_z$. In fact, the evaluations $f \circ (g \circ h)$ and $(f \circ g) \circ h$ of $f(gh)$ and $(fg)h$ in $\mathcal{C}$ may differ.

Now let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism. Let $w$ be a non-associative word in $\mathcal{C}$. It defines a non-associative word $F(w)$ in $\mathcal{D}$ by replacing an arrow $f \in \mathcal{C}$ by $F(f)$ and $I_x$ for $x \in \mathcal{C}^0$ by $I_{F^0(x)}$.

LEMMA 4.8.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism. The 2-arrows $\mu$ and $\lambda$ in the data of $F$ may be combined in a unique way to give well defined 2-arrows*

$$\Phi_w\colon \mathrm{ev}\bigl(F(w)\bigr) \Rightarrow F\bigl(\mathrm{ev}(w)\bigr)$$

*for all non-associative words $w$. These are invertible if $F$ is a homomorphism.*

PROOF. The 2-arrows $\Phi_w$ are defined by a recursion over the complexity of words. The basic cases are words of the form $I_x$ for $x \in \mathcal{C}^0$ and $fg$ for composable arrows in $\mathcal{C}$. In the first case, $\mathrm{ev}\bigl(F(I_x)\bigr) = 1_{F^0(x)}$ and $F\bigl(\mathrm{ev}(I_x)\bigr) = F(1_x)$, and

$$\Phi_{I_x} := \lambda_x\colon 1_{F^0(x)} \Rightarrow F(1_x)$$

works. In the second case, we let

$$\Phi_{fg} = \mu_{f,g}\colon \mathrm{ev} \circ F(fg) = F(f) \circ F(g) \Rightarrow F(f \circ g) = F \circ \mathrm{ev}(fg).$$

Now consider an arbitrary word $w$. Let $w'$ be the word where each letter of the form $I_x$ for some $x \in \mathcal{C}^0$ is replaced by $1_x$. Then $\mathrm{ev}(w) = \mathrm{ev}(w')$. And $\mathrm{ev} \circ F(w)$ and $F \circ \mathrm{ev}(w')$ differ in that factors $1_{F^0(x)}$ are replaced by $F(1_x)$, whereas factors $F(f)$ for arrows $f \in \mathcal{C}$ remain the same, and the parentheses are also still the same. A horizontal product of the 2-arrows $\Phi_{I_x}$ for letters $I_x$ in $w$ and the unit 2-arrows $1_f$ for letters $f$ in $w$ gives a 2-arrow $\Phi_{w,w'}\colon \mathrm{ev}(F(w)) \Rightarrow \mathrm{ev}(F(w'))$. We could also construct this 2-arrow by replacing one letter $I_x$ by $1_x$ at a time. This would give the same 2-arrow because horizontal and vertical products in a bicategory commute.

The word $w'$ no longer contains letters of the form $I_x$. It must contain the combination $fg$ not broken by parentheses at least once, perhaps many times. Let $w''$ be the word where each combination $fg$ in $w$ is replaced by its evaluation $f \circ g \in \mathcal{C}$, which is now a single letter, and unnecessary parentheses are removed. For instance, if $w' = (fg)(hk)$, then $w''$ is the 2-letter word $w'' = f \circ g\ h \circ k$ without any parentheses. The evaluations $\mathrm{ev}(w')$ and $\mathrm{ev}(w'')$ are equal by construction. The horizontal product of the 2-arrows $\Phi_{f,g}$ for all unbroken composable pairs $fg$ in $w$ and unit 2-arrows $1_h$ for other leters $h$ gives a 2-arrow $\Phi_{w',w''} \colon \mathrm{ev}(F(w')) \Rightarrow \mathrm{ev}(F(w''))$. Now we repeat the process above to simplify $w''$ further, until we arrive at a one-letter word. Then $\mathrm{ev}(F(f)) = F(\mathrm{ev}(f))$ and we are done. This process terminates because the nesting of parentheses in $w''$ is one less than for $w'$. The 2-arrows above are all built from the 2-arrows $\lambda_x$ and $\mu_{f,g}$ in the definition of a morphism. If the latter are invertible, so are horizontal products of them with more 2-arrows of this form or with unit 2-arrows. Therefore, the 2-arrows $\Phi_w$ above are invertible if $F$ is a morphism.

If $w_1$ and $w_2$ are two words, then so are $(w_1)(w_2)$; here we leave out the parentheses around $w_1$ or $w_2$ if that word has length 1. The evaluation satisfies $\mathrm{ev}((w_1)(w_2)) = \mathrm{ev}(w_1) \circ \mathrm{ev}(w_2)$. The recursive definition of $\Phi_w$ implies that the following diagram of 2-arrows commutes:

$$\begin{array}{ccc} \mathrm{ev} \circ F\big((w_1)(w_2)\big) & = & \big(\mathrm{ev} \circ F(w_1)\big) \circ \big(\mathrm{ev} \circ F(w_2)\big) \\ \Downarrow{\scriptstyle \Phi_{(w_1)(w_2)}} & & \Downarrow{\scriptstyle \Phi_{w_1} \bullet \Phi_{w_2}} \\ F \circ \mathrm{ev}\big((w_1)(w_2)\big) & \overset{}{\underset{\mu_{\mathrm{ev}(w_1),\mathrm{ev}(w_2)}}{\Leftarrow}} & \big(F \circ \mathrm{ev}(w_1)\big) \circ \big(F \circ \mathrm{ev}(w_2)\big) \end{array}$$

Similarly, if $w$ is a word and the source and range of its first and last letter are $y$ and $x$, then there are words $I_y(w)$ and $(w)I_x$, and there are identities of 2-arrows

$$\begin{array}{ccc} \mathrm{ev} \circ F(I_y(w)) & = & 1_{F^0(y)} \circ \big(\mathrm{ev} \circ F(w)\big) \\ \Downarrow{\scriptstyle \Phi_{I_y(w)}} & & \Downarrow{\scriptstyle \lambda_y \bullet \Phi_w} \\ F \circ \mathrm{ev}(I_y(w)) & = & F(1_y) \circ \big(F \circ \mathrm{ev}(w)\big) \end{array}$$

and similarly for $(w)I_x$. These facts imply by a recursive argument that any 2-arrow $\mathrm{ev}\big(F(w)\big) \Rightarrow F\big(\mathrm{ev}(w)\big)$ built out of the 2-arrows $\mu^F$ and $\lambda$ by horizontal and vertical products is equal to $\Phi_w$. □

An associator in $\mathcal{C}$ gives us a 2-arrow between the evaluations of non-associative words where one pair of parentheses is shifted, replacing $(w_1w_2)w_3$ for words $w_1, w_2, w_3$ by $w_1(w_2w_3)$. And a uniter in $\mathcal{C}$ gives a 2-arrow between the evaluations of non-associative words where one letter $I_x$ is deleted. Let $w_1$ and $w_2$ be non-associative words that are related like this, and let $\alpha \colon \mathrm{ev}(w_1) \Rightarrow \mathrm{ev}(w_2)$ be the associator or uniter that links them. Then $F(\alpha)$ is a 2-arrow $F \circ \mathrm{ev}(w_1) \Rightarrow F \circ \mathrm{ev}(w_2)$. We may also first apply $F$ and then evaluate. Then a suitable associator or uniter in $\mathcal{D}$ gives a 2-arrow $\alpha' \colon \mathrm{ev} \circ F(w_1) \Rightarrow \mathrm{ev} \circ F(w_2)$. We claim that in this situation, the following diagram of 2-arrows commutes:

$$\begin{array}{ccc} \mathrm{ev} \circ F(w_1) & \overset{\Phi_{w_1}}{\Longrightarrow} & F \circ \mathrm{ev}(w_1) \\ {\scriptstyle \alpha'}\Downarrow & & \Downarrow{\scriptstyle F(\alpha)} \\ \mathrm{ev} \circ F(w_2) & \overset{\Phi_{w_2}}{\Longrightarrow} & F \circ \mathrm{ev}(w_2) \end{array} \tag{4.8.1}$$

We briefly say that $\Phi$ intertwines associators and uniters in $\mathcal{C}$ and $\mathcal{D}$.

Because of the recursive definition of $\Phi$, it suffices to prove the claim in the three basic cases

$$\alpha = l_f \colon \mathrm{ev}(I_y f) \Rightarrow \mathrm{ev}(f),$$
$$\alpha = r_f \colon \mathrm{ev}(f I_x) \Rightarrow \mathrm{ev}(f),$$
$$\alpha = \mathrm{ass} \colon \mathrm{ev}((f_1 f_2) f_3) \Rightarrow \mathrm{ev}(f_1(f_2 f_3)).$$

The diagrams (4.8.1) for these three cases commute because they are exactly the three coherence conditions for a morphism of bicategories in (4.3.1) and (4.3.2).

Now suppose that one non-associative word $w_1$ is transformed into another word $w_2$ using several associators, uniters and their inverses. This defines a 2-arrow $A\colon \mathrm{ev}(w_1) \Rightarrow \mathrm{ev}(w_2)$. Replacing each associator or uniter $\alpha$ in $\mathcal{C}$ by the corresponding associator or uniter $\alpha'$ in $\mathcal{D}$, we get a corresponding 2-arrow $A'\colon \mathrm{ev}(F(w_1)) \Rightarrow \mathrm{ev}(F(w_2))$. The commuting diagrams (4.8.1) for all factors $\alpha^{\pm 1}$ in $A$ imply that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{ev} \circ F(w_1) & \xRightarrow{\Phi_{w_1}} & F \circ \mathrm{ev}(w_1) \\ {\scriptstyle A'}\Downarrow & & \Downarrow{\scriptstyle F(A)} \\ \mathrm{ev} \circ F(w_2) & \xRightarrow{\Phi_{w_2}} & F \circ \mathrm{ev}(w_2) \end{array} \tag{4.8.2}$$

THEOREM 4.8.2. *Let $\mathcal{C}$ be a bicategory. Let $w_1, w_2$ be non-associative words in $\mathcal{C}$ and let $A_1, A_2\colon \mathrm{ev}(w_1) \Rightarrow \mathrm{ev}(w_2)$ be two 2-arrows that are built out of associators and uniters and their inverses. Then $A_1 = A_2$.*

PROOF. We use the Yoneda homomorphism $\mathbb{Y}\colon \mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat})$. This is a homomorphism from $\mathcal{C}$ to $\mathcal{D} := \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat})$. Since $\mathcal{D}$ is strict and the 2-arrows $A_1', A_2'\colon \mathrm{ev} \circ \mathbb{Y}(w_1) \Rightarrow \mathrm{ev} \circ \mathbb{Y}(w_2)$ associated to $A_1$ and $A_2$ are built out of associators and uniters in $\mathcal{D}$, they are both unit 2-arrows. Then the commuting diagram (4.8.2) implies $\mathbb{Y}(A_2) \circ \Phi_{w_1} = \mathbb{Y}(A_1) \circ \Phi_{w_1}$. Since $\mathbb{Y}$ is a homomorphism, $\Phi_{w_1}$ is invertible by Lemma 4.8.1. Then $\mathbb{Y}(A_2) = \mathbb{Y}(A_1)$ follows. Since $\mathbb{Y}$ is an equivalence of categories from $\mathcal{C}(x,y)$ to $\mathcal{D}(\mathbb{Y}(x), \mathbb{Y}(y))$ for all $x, y \in \mathcal{C}^0$, it is fully faithful on 2-arrows. Then $A_1 = A_2$ follows. □

Theorem 4.8.2 makes precise which coherence conditions to expect from the weakening scheme: the pairs of 2-arrows $A_1, A_2$ in it are exactly what it means to have two ways to prove an identity in a 2-category in two different ways.

## 4.9. Strictification of bicategories and classification of bigroups

The proof of the Coherence Theorem for bicategories using the Yoneda Embedding also implies more, namely, that any bicatgegory is "equivalent" to a strict one. Here there are two ways to define equivalent. We first discuss the easier one, which is based on a 2-category of bicategories with morphisms as arrows and icons as 2-arrows; this is due to Lack [**15**]. Then we apply it to clarify the classification of crossed modules.

We have already defined a product of morphisms in Section 4.7.3 and seen that this gives a category. To produce a more interesting concept of equivalence, we must add appropriate 2-arrows to this category. The easiest choice are icons. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be bicategories. Let $G_1, G_2\colon \mathcal{C} \rightrightarrows \mathcal{D}$ and $F\colon \mathcal{D} \rightrightarrows \mathcal{E}$ be pairs of parallel morphisms that are equal on objects. Let $\alpha\colon F_1 \Rightarrow F_2$ and $\beta\colon G_1 \Rightarrow G_2$ be icons (see Definition 4.3.12). These consist of natural 2-arrows $\alpha_f\colon F_1(f) \Rightarrow F_2(f)$ for all arrows $f \in \mathcal{D}$ and $\beta_f\colon G_1(g) \Rightarrow G_2(g)$ for all $g \in \mathcal{C}$, such that certain diagrams of 2-arrows commute. By definition, $(F_j * G_j)(g) = F_j(G_j(g))$. The diagram of natural 2-arrows

$$\begin{array}{ccc} F_1(G_1(g)) & \xRightarrow{\alpha_{G_1(g)}} & F_2(G_1(g)) \\ {\scriptstyle F_1(\beta_g)}\Downarrow & & \Downarrow{\scriptstyle F_2(\beta_g)} \\ F_1(G_2(g)) & \xRightarrow{\alpha_{G_2(g)}} & F_2(G_2(g)) \end{array}$$

commutes because $\alpha$ is natural relative to $\beta_g$. Define

$$(\alpha * \beta)_f := \alpha_{G_2(g)} \cdot F_1(\beta_g) = F_2(\beta_g) \cdot \alpha_{G_1(g)} \colon F_1(G_1(g)) \Rightarrow F_2(G_2(g)).$$

PROPOSITION 4.9.1. *These* 2*-arrows form an icon* $(\alpha * \beta)_f \colon F_1 * G_1 \Rightarrow F_2 * G_2$*. This is the horizontal product of a strict* 2*-category with bicategories as objects, morphisms as arrows, icons as* 2*-arrows, the product of arrows* $*$*, and the obvious vertical product of icons. We call this bicategory the* icon 2-category*.*

PROOF. All this is elementary to check and left as an exercise. □

THEOREM 4.9.2. *A morphism* $F \colon \mathcal{C} \to \mathcal{D}$ *is an equivalence in the icon* 2*-category if and only if it is bijective on objects, a homomorphism, and the functors* $\mathcal{C}(x, y) \to \mathcal{D}(F^0(x), F^0(y))$ *are equivalences of categories for all* $x, y \in \mathcal{C}^0$*.*

PROOF. Assume first that $F$ is an equivalence in the icon 2-category. Let $G \colon \mathcal{D} \to \mathcal{C}$ be an inverse equivalence. So there must be invertible icons $G * F \Rightarrow \mathrm{id}_\mathcal{C}$ and $F * G \Rightarrow \mathrm{id}_\mathcal{D}$. Then $G * F$ and $\mathrm{id}_\mathcal{C}$ are equal on objects, and so are $F * G$ and $\mathrm{id}_\mathcal{D}$. Thus $F^0$ and $G^0$ are bijections inverse to each other. And if $x, y \in \mathcal{C}^0$, then the invertible icon $G * F \Rightarrow \mathrm{id}_\mathcal{C}$ gives a natural isomorphism between the composite functor

$$\mathcal{C}(x, y) \xrightarrow{F} \mathcal{D}(F^0(x), F^0(y)) \xrightarrow{G} \mathcal{C}(x, y)$$

and the identity functor, and similarly for the composite functor

$$\mathcal{D}(F^0(x), F^0(y)) \xrightarrow{G} \mathcal{C}(x, y) \xrightarrow{F} \mathcal{D}(F^0(x), F^0(y)).$$

So these functors are equivalences of categories that are inverse to each other. In the diagrams in Definition 4.3.12 for the invertible icon $F * G \Rightarrow \mathrm{id}_\mathcal{D}$, the vertical arrows are invertible by assumption, and the bottom horizontal arrows are identity maps; therefore, $\mu^{F*G}$ and $\lambda^{F*G}$ are invertible. So are $\mu^{G*F}$ and $\lambda^{G*F}$. This implies that $\mu^G_{F(g),F(f)}$ and $\lambda^G_{F^0(x)}$ for composable arrows $(g, f)$ in $\mathcal{C}$ and $x \in \mathcal{C}^0$ and $\mu^F_{G(k),G(h)}$ and $\lambda^F_{G^0(y)}$ for composable arrows $(k, h)$ in $\mathcal{D}$ and $x \in \mathcal{D}^0$ are right invertible, whereas $G(\mu^F_{g,f})$ and $G(\lambda^F_x)$ and $F(\mu^G_{k,h})$ and $F(\lambda^G_y)$ are left invertible. Since $G$ is an equivalence of categories, it preserves and respects invertibility. So already $\mu^F_{g,f}$ and $\lambda^F_x$ are left invertible. Any pair of composable arrows $(g, f)$ in $\mathcal{C}$ is isomorphic to one of the form $G(k), G(h)$ and any object is equal to $G^0(y)$ for some $y$. So $\mu^F_{g,f}$ and $\lambda^F_x$ are both left and right invertible. This means that $F$ is a homomorphism.

Now assume, conversely, that $F \colon \mathcal{C} \to \mathcal{D}$ is a homomorphism that is bijective on objects and contains equivalences of categories $F_{x,y} \colon \mathcal{C}(x, y) \to \mathcal{D}(F^0(x), F^0(y))$ for all $x, y \in \mathcal{C}^0$. Let $G^0 \colon \mathcal{D}^0 \to \mathcal{C}^0$ be the inverse of the bijection $F^0$ on objects. For $z, w \in \mathcal{D}^0$, there are a functor $G_{z,w} \colon \mathcal{D}(z, w) \to \mathcal{C}(G^0(z), G^0(w))$ and natural isomorphisms

$$\varepsilon \colon F_{G^0(z),G^0(w)} \circ G_{z,w} \simeq \mathrm{id}_{\mathcal{D}(z,w)}, \qquad \eta \colon G_{z,w} \circ F_{G^0(z),G^0(w)} \simeq \mathrm{id}_{\mathcal{C}(G^0(z),G^0(w))}$$

that form an adjoint equivalence of functors (see Exercise 4.2.9). We want to embed the maps $G^0$ on objects and the functors $G_{z,w}$ into a homomorphism $\mathcal{D} \to \mathcal{C}$. We shall briefly write $F$ and $G$ without indices in the following. We need 2-arrows $\mu^G_{g,h} \colon G(g) \circ G(h) \Rightarrow G(g \circ h)$ for composable $g \in \mathcal{D}(z, w)$, $h \in \mathcal{D}(y, z)$ and $\lambda^G_x \colon 1_{G^0(y)} \Rightarrow G(1_y)$ for $y \in \mathcal{D}^0$. Due to the adjoint equivalence, there is a bijection between 2-arrows $G(g) \circ G(h) \Rightarrow G(g \circ h)$ in $\mathcal{C}$ and 2-arrows $F(G(g) \circ G(h)) \Rightarrow g \circ h$ in $\mathcal{D}$. We let $\mu^G_{g,h}$ be the adjunct of the 2-arrow

$$F(G(g) \circ G(h)) \xRightarrow{(\mu^F_{G(g),G(h)})^{-1}} F(G(g)) \circ F(G(h)) \xRightarrow{\varepsilon_g \circ \varepsilon_h} g \circ h.$$

Similarly, there is a bijection between 2-arrows $1_{G^0(y)} \Rightarrow G(1_y)$ and $F(1_{G^0(y)}) \Rightarrow 1_y$, and we let $\lambda^G_x$ be the adjunct of the 2-arrow

$$F(1_{G^0(y)}) \xRightarrow{(\lambda^F_{G^0(y)})^{-1}} 1_{F^0(G^0(y))} = 1_y.$$

The 2-arrows $F(G(g) \circ G(h)) \Rightarrow g \circ h$ above are invertible and natural for 2-arrows $g_1 \Rightarrow g_2$, $h_1 \Rightarrow h_2$. This is inherited by the 2-arrows $\mu^G_{g,h}$. We leave it to the reader to check that the coherence diagrams for a homomorphism in Definition 4.3.1 commute; using the adjoint equivalence, this is reduced to the corresponding statement for $F$. The homomorphisms $F * G$ and $G * F$ are the identity maps on objects, and they act by $F \circ G$ and $G \circ F$ on arrows. The natural isomorphisms $\varepsilon$ and $\eta$ above provide natural 2-arrows $F \circ G(g) \Rightarrow g$ and $G \circ F(f) \Rightarrow f$ for arrows $f$ in $\mathcal{C}$ and $g$ in $\mathcal{D}$. These form invertible icons, that is, the diagrams in Definition 4.3.12 commute. Thus $F$ is an equivalence in the icon 2-category with inverse $G$. □

**Add Lemma saying that** $\mathrm{Mor}(\mathcal{C}, \mathcal{B})$ **and** $\mathrm{Hom}(\mathcal{C}, Cat[B])$ **is strict if** $\mathcal{B}$ **is strict and cite it here.**

THEOREM 4.9.3. *Let $\mathcal{C}$ be a bicategory. Let $\mathcal{C}_2 \subseteq \mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat})$ be the sub-2-category that has $\mathbb{Y}(c)$ for $c \in \mathcal{C}^0$ as objects, all strong transformations $\mathbb{Y}(c) \Rightarrow \mathbb{Y}(d)$ as arrows, and all modifications between these as 2-arrows. The Yoneda Embedding $\mathbb{Y}\colon \mathcal{C} \to \mathcal{C}_2$ is an equivalence in the icon 2-category.*

PROOF. If $\mathbb{Y}(c) = \mathbb{Y}(c')$, then $c = c'$. Thus the map $\mathcal{C} \to \mathcal{C}_2$ is bijective on objects. The functors $\mathcal{C}(x, y) \to \mathrm{Hom}(\mathbb{Y}(x), \mathbb{Y}(y))$ for $x, y \in \mathcal{C}^0$ are equivalences of categories by Corollary 4.7.6. Then the claim follows from Theorem 4.9.2. □

Theorem 4.9.3 shows that any bicategory is equivalent to a strict one, namely, the 2-category $\mathcal{C}_2$. We call it the *Yoneda strictification* of $\mathcal{C}$. It follows that most results about 2-categories can be extended to bicategories.

EXAMPLE 4.9.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, viewed as bicategories with only unit 2-arrows. Then an equivalence between $\mathcal{C}$ and $\mathcal{D}$ in the icon 2-category is already an isomorphism of categories by Theorem 4.9.2. This example shows the need for a weaker concept of equivalence of bicategories that generalises equivalence of categories. We will introduce this concept in Section 4.10.

Our next goal is to classify bigroups up to equivalence.

DEFINITION 4.9.5. A *bigroup* is a bicategory with one object and such that each arrow is an equivalence and each 2-arrow is invertible.

THEOREM 4.9.6. *Any bigroup is equivalent to a bigroup with the property that two arrows that are isomorphic are already equal. Then each arrow is invertible and the unit is strict. A bigroup with these extra properties is pinned down by the following data:*

- *the group $G$ of arrows with product;*
- *the Abelian group $H$ of 2-arrows $1 \Rightarrow 1$ with the vertical product, which is equal to the horizontal product;*
- *the $G$-action on $H$ by whiskering, $c_g(h) := 1_g \bullet h \bullet 1_{g^{-1}}$;*
- *the associator $G \times G \times G \to H$.*

*Two bigroups described in this way are equivalent in the icon 2-category if and only if there are group isomorphisms $G_1 \cong G_2$ and $H_1 \cong H_2$ that preserve the $G_j$-actions on $H_j$ and turn the associators into each other up to a coboundary.*

PROOF. Let $\mathcal{C}$ be a bigroup. Let $c$ be its unique object and let $\mathcal{C}(c,c)$ be the category with arrows $c \to c$ as objects and 2-arrows between these as arrows. This is a groupoid by assumption. Let $G \subseteq \mathcal{C}(c,c)^0$ be a subset that contains exactly one representative from each isomorphism class of arrows. We assume also that $1_c \in G$. For each $f \in \mathcal{C}(c,c)^0$, let $P(f) \in G$ be the representative from the orbit of $f$ and let $\alpha_f \colon f \Rightarrow P(f)$ be a 2-arrow, necessarily invertible. We assume that $\alpha_{1_c} = 1_{1_c}$. There is a unique way to extend $P$ to a functor $P \colon \mathcal{C}(c,c) \to \mathcal{C}(c,c)$ such that $\alpha$ is a natural transformation $\mathrm{id}_{\mathcal{C}(c,c)} \Rightarrow P$. Namely, if $f_1, f_2 \in \mathcal{C}(c,c)$, $b \colon f_1 \Rightarrow f_2$, then

$$P(b) = \alpha_{f_2} \cdot b \cdot \alpha_{f_1}^{-1} \colon P(f_1) \xrightarrow{\alpha_{f_1}^{-1}} f_1 \xRightarrow{b} f_2 \xrightarrow{\alpha_{f_2}} P(f_2).$$

There is a unique way to turn $P$ into a homomorphism $\mathcal{C} \to \mathcal{C}$ such that $\alpha$ is an icon $\mathrm{id}_{\mathcal{C}} \to P$. Namely, if $(f,g) \in \mathcal{C}$ are composable, then we let

$$\mu^P_{f,g} := \alpha_{f \circ g} \circ (\alpha_f^{-1} \bullet \alpha_g^{-1}), \qquad \lambda^P_c := \alpha_{1_c} = 1_{1_c}.$$

These 2-arrows are invertible and the diagrams in Definition 4.3.12 commute by construction. An elementary calculation shows that $P$ defined like this is a homomorphism. By construction, it is equivalent to the identity homomorphism in the icon 2-category.

Now we want to turn the image of $P$ into a bigroup $\mathcal{D}$ in its own right, so that $P$ becomes a homomorphism onto it. The image of $P$ on arrows is the set $\mathcal{D}(c,c)^0 := G \subseteq \mathcal{C}(c,c)^0$. Its image on 2-arrows is the set of all 2-arrows among arrows in $G$. Define the product $\circ_{\mathcal{D}}$ by

$$f \circ_{\mathcal{D}} g := P(f \circ g) \in G.$$

The associator in $\mathcal{C}$ gives invertible 2-arrows $(f \circ g) \circ h \cong f \circ (g \circ h)$ for $(f,g,h) \in G^3$. This implies $P\big((f \circ g) \circ h\big) = P\big(f \circ (g \circ h)\big)$. That is, the multiplication $\circ_{\mathcal{D}}$ is associative on $G$. Similarly, $1_c \in G$ is a unit for it and each element of $G$ has an inverse, making it a group. The vertical product in $\mathcal{C}$ restricts to a vertical product in $\mathcal{D}$. The horizontal product is defined by $a \bullet_{\mathcal{D}} b := P(a \bullet b)$. By construction, if there is a 2-arrow $a \colon f \Rightarrow g$ in $G$, then $a$ is invertible and $f = g$. If $(f,g,h) \in \mathcal{C}^3$ are three composable arrows, then there is a unique associator

$$\mathrm{ass}_{\mathcal{D}} \colon (f \circ_{\mathcal{D}} g) \circ_{\mathcal{D}} h \Rightarrow f \circ_{\mathcal{D}} (g \circ_{\mathcal{D}} h)$$

that makes the diagram (4.3.1) commute for our would-be homomorphism $P \colon \mathcal{C} \to \mathcal{D}$. These associators may be nontrivial although $\circ_{\mathcal{D}}$ is already associative. Similarly, there are unique left and right uniters in $\mathcal{D}$ that make the two diagrams in (4.3.2) commute. We equip $\mathcal{D}$ with this extra structure. It is routine to check that this defines a bicategory, that is, the associators and uniters in $\mathcal{D}$ inherit the necessary properties from $\mathcal{C}$.

So far, we have seen that any bigroup $\mathcal{C}$ is equivalent to a bigroup $\mathcal{D}$ with the property that $f = g$ whenever there is a 2-arrow $f \Rightarrow g$. Conversely, assume that $\mathcal{C}$ itself already has this property. Then $(f \circ g) \circ h = f \circ (g \circ h)$ and $1_c \circ f = f \circ 1_c$ for all $f, g, h \in G$. So $G$ becomes a group. Since any arrow is invertible, any 2-arrow $f \Rightarrow f$ is of the form $1_f \bullet a$ for a 2-arrow $a \colon 1_c \Rightarrow 1_c$. The 2-arrows $1_c \Rightarrow 1_c$ form a group $H$ both under the horizontal and the vertical products. These two products must be equal and commutative (compare the computations for strict 2-categories in Section 2.6.1, in the special case where $\partial$ is trivial). So $H$ is an Abelian group. Since $(1_f \bullet a) \cdot (1_f \bullet b) = 1_f \bullet (a \cdot b)$, the vertical multiplication of 2-arrows $f \Rightarrow f$ is commutative as well. Then the naturality of the associators in (4.2.2) says simply that the horizontal product on the 2-arrows of $\mathcal{C}$ is associative: $(c_1 \bullet c_2) \bullet c_3 = c_1 \bullet (c_2 \bullet c_3)$. Then the 2-arrows of $\mathcal{C}$ with the horizontal product form a group $K$. Therefore, there is a strict 2-group $\mathcal{C}'$ that has the same arrows and 2-arrows, the same product of arrows and the same horizontal and vertical products of 2-arrows – only the associators and uniters are replaced by trivial ones. This strict 2-category is described by a crossed module as in Section 2.6.1. So $\mathcal{C}$ is described uniquely by this crossed module and by its associators and left and right uniters.

Next, we arrange for the two uniters to be trivial by applying an equivalence in the icon 2-category that is based on the identity functor $\mathcal{C}(c, c) \to \mathcal{C}(c, c)$ and $\lambda_c = 1_{1_c}$. We choose nontrivial $\mu_{f,g}\colon f \circ g \Rightarrow f \circ g$, however. Then (4.3.1) and (4.3.2) dictate how to change the associators and uniters in $\mathcal{C}$ to make this a homomorphism. In particular, we see that the new right and left uniters are $r_f \circ \mu_{f,1_c}$ and $l_f \circ \mu_{1_c,f}$, respectively. So choosing $\mu_{f,1_c} := r_f^{-1}$ and $\mu_{1_c,f} := l_f^{-1}$, we find an invertible homomorphism between $\mathcal{C}$ and another bigroup that has trivial uniters. So only the crossed module $\mathcal{C}'$ and the (changed) associator remain as invariants. Writing $\mathrm{ass}_{f,g,h} = \omega_{f,g,h} \bullet 1_{f \circ g \circ h}$ with $\omega_{f,g,h}\colon 1_c \Rightarrow 1_c$, we describe the associators through a map $G^3 \to H$, $(f, g, h) \mapsto \omega_{f,g,h}$. The commuting diagrams (4.2.3) and (4.2.5) now say that the associator is normalised, that is, $\omega_{1,f_1,f_2} = 1$, $\omega_{f_1,1,f_2} = 1$, and $\omega_{f_1,f_2,1} = 1$ for all arrows $f_1, f_2 \in G$. And (4.2.4) says that the associator satisfies the cocycle condition

$$\omega_{f_1,f_2,f_3 \cdot f_4} \omega_{f_1 f_2,f_3,f_4} = c_{f_1}(\omega_{f_2,f_3,f_4}) \omega_{f_1,f_2 f_3,f_4} \omega_{f_2,f_3,f_4}$$

for all $f_1, f_2, f_3, f_4 \in G$. In other words, $\omega$ is a normalised 3-cocycle $G^3 \to H$. We may still modify the associator using an equivalence as above, based on the identity functor and some 2-arrows $\mu_{f,g}$. We must put $\mu_{1,g} = 1$ and $\mu_{f,1} = 1$ in order to leave the uniters unchanged. And the condition (4.3.1) for a morphism dictates that this equivalence replaces the associator $\omega$ by $\omega \cdot \partial(\mu^{-1})$ with

$$(\partial\mu^{-1})_{f,g,h} := \mu_{f,g} \mu_{fg,h} \mu^{-1}_{f,gh} c_f(\mu_{g,h})$$

for all $f, g, h \in G$. Thus $\omega$ may be modified by an arbitrary coboundary, and the cohomology class of $\omega$ remains as an invariant.

Now let $\mathcal{C}$ and $\mathcal{C}'$ be two bigroups with all the extra properties arranged above, that is, $f = g$ if there is a 2-arrow $f \Rightarrow g$ and the uniters are trivial. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an equivalence in the icon 2-category. This means that $F$ is a homomorphism and that the functor $\mathcal{C}(c, c) \to \mathcal{C}'(c, c)$ is an equivalence of categories. Since all arrows in these two categories have the same range and souce, one checks that this equivalence must already be an isomorphism. Therefore, $F$ induces an isomorphism between the crossed modules associated to $\mathcal{C}$ and $\mathcal{C}'$ by ignoring their associators. There is an invertible transformation between $F$ and a strictly unital homomorphism: this amounts to choosing $\sigma_{1_c} = \lambda_c$ (compare Exercise 2.4.2 for a similar construction for homomorphisms to $\mathcal{C}^*(2)$). Therefore, we may assume without loss of generality that $F$ is already strictly unital. Since both $\mathcal{C}$ and $\mathcal{C}'$ have trivial uniters, it follows that $\mu^F_{f,1} = 1$ and $\mu^F_{1,f} = 1$ for all arrows $f \in \mathcal{C}$. Then the computations above show that $F$ modifies the associators by a coboundary. So an equivalence between $\mathcal{C}$ and $\mathcal{C}'$ in the icon 2-category exists if and only if there is an isomorphism between the crossed modules underlying $\mathcal{C}$ and $\mathcal{C}'$ that preserves the cohomology classes of the associators. □

COROLLARY 4.9.7. *Let $\mathcal{C}$ be a bigroup with the property that $f = g$ whenever $a\colon f \Rightarrow g$ is a 2-arrow in $\mathcal{C}$. Then the Yoneda strictification $\mathcal{C}_2$ is a strict 2-group. Thus any bigroup is equivalent to a strict 2-group in the icon 2-category.*

PROOF. By definition, $\mathcal{C}_2$ has exactly one object, namely, $\mathbb{Y}(c)$ for the unique object $c \in \mathcal{C}^0$. And $\mathcal{C}_2$ is a strict 2-category by construction. Equivalences in the icon 2-category preserve the property that all 2-arrows are invertible because of Theorem 4.9.2. A strong transformation $\mathbb{Y}(c) \Rightarrow \mathbb{Y}(c)$ consists of a functor $\sigma_c : \mathcal{C}(c,c) \to \mathcal{C}(c,c)$ and natural isomorphisms of functors $\sigma_f : \mathbb{Y}(f) \circ \sigma_c \Rightarrow \sigma_c \circ \mathbb{Y}(f)$ for all arrows $f \in \mathcal{C}(c,c)^0$, subject to some conditions. Since all arrows in $\mathcal{C}$ are equivalences and $\mathcal{C}_2$ is equivalent to $\mathcal{C}$, the functor $\sigma_c$ must be an equivalence of categories. Since all arrows in $\mathcal{C}(c,c)$ have the same source and target, this forces $\sigma_c$ to be an isomorphism of categories. So $\sigma$ is invertible. This shows that $\mathcal{C}_2$ is a strict 2-group. □

The results above provide another perspective on the classification of crossed modules in Section 2.8. Recall that this classification uses a primary invariant and a secondary invariant. The primary invariant consists of the group $\pi_1(\mathcal{C})$ and the Abelian group $\pi_2(\mathcal{C})$, together with an action of $\pi_1(\mathcal{C})$ on $\pi_2(\mathcal{C})$ by automorphisms. The secondary invariant is the MacLane–Whitehead obstruction in the third cohomology group $\mathrm{H}^3(\pi_1(\mathcal{C}), \pi_2(\mathcal{C}))$ in Theorem 2.8.8. Crossed modules are equivalent to strict 2-groups (see Section 2.6.1). By Corollary 4.9.7, any bigroup is equivalent to a strict 2-group. The primary invariant for strict 2-groups makes sense for bigroups as well. Namely, we have associated to any bigroup a strict 2-group with the extra property that $f = g$ whenever there is a 2-arrow $f \Rightarrow g$. Such strict 2-groups are equivalent to crossed modules with the extra property that $\partial$ is trivial. And such crossed modules are the same as the primary invariants of crossed modules. In addition, Theorem 4.9.6 says that bigroups with fixed primary invariant are classified by a third cohomology class, namely, the class of the associator. Here any third cohomology class occurs. And Corollary 4.9.7 shows that any third cohomology class is realised by some strict 2-group. The classification theorem in Section 2.8 used a concept of equivalence of crossed modules that was generated by "elementary equivalences". When we translate the latter to strict 2-groups, then elementary equivalences are simply *strict* homomorphisms that are equivalences. So elementary equivalence is defined using zigzags of strict homomorphisms that are equivalences. Combining the classification results for crossed modules up to equivalence of crossed modules in Theorem 2.8.8 and for bigroups in Theorem 4.9.6, we conclude that two crossed modules are equivalent as crossed modules if and only if the associated strict 2-groups are equivalent in the icon 2-category. This is remarkable because the latter equivalences are defined directly, without zigzags. The reason why we do not need zigzags in the icon 2-category is Theorem 4.9.2, which says that any homomorphism that deserves to be an equivalence already has an inverse. This allows to collapse zigzags to a single equivalence. In order to deduce the classification in Theorem 2.8.8 from the general results in bicategories, we would need also a Coherence Theorem for homomorphisms, which replaces them by strict homomorphisms in a sufficiently nice way.

The following exercise generalises the results for bigroups and 2-groups above to more general bicategories. The proofs are the same as above. But there are several objects and there may be non-invertible arrows and 2-arrows.

EXERCISE 4.9.8. *Let $\mathcal{C}$ be a bicategory. Show that $\mathcal{C}$ is equivalent in the icon 2-category to a bicategory $\mathcal{C}'$ where isomorphic arrows are equal. Show that the Yoneda strictification of $\mathcal{C}'$ is a 2-category with the property that any equivalence in $\mathcal{C}'$ is already an isomorphism.*

## 4.10. Equivalence of bicategories

**4.10.1. More compositions.** Let $G_1, G_2 \colon \mathcal{C} \rightrightarrows \mathcal{D}$ and $F_1, F_2 \colon \mathcal{D} \rightrightarrows \mathcal{E}$ be parallel pairs of morphisms. Let $\tau \colon G_1 \Rightarrow G_2$ and $\sigma \colon F_1 \Rightarrow F_2$ be transformations. In this generality, we cannot define a product transformation from $F_1 * G_1$ to $F_2 * G_2$: we need $F_1$ and $F_2$ to be homomorphisms. And under this assumption, there are two ways to define a product transformation $F_1 * G_1 \Rightarrow F_2 * G_2$. These are only equal if $\sigma$ and $\tau$ are strong transformations. It is useful to reduce the complexity of the situation by assuming that $\tau$ or $\sigma$ is a unit transformation. This is no loss of generality because we would expect a commuting square of transformations

$$\begin{array}{ccc} F_1 * G_1 & \overset{\sigma \square G_1}{\Longrightarrow} & F_2 * G_1 \\ {\scriptstyle F_1 \square \tau}\Downarrow & \overset{\sigma \square \tau}{\searrow} & \Downarrow{\scriptstyle F_2 \square \tau} \\ F_1 * G_2 & \underset{\sigma \square G_2}{\Longrightarrow} & F_2 * G_2. \end{array}$$

However, if $\tau$ and $\sigma$ are general transformations, then the square above does not commute. Instead, there is a modification

$$(F_2 \,\square\, \tau) \circ (\sigma \,\square\, G_1) \Rrightarrow (\sigma \,\square\, G_2) \circ (F_1 \,\square\, \tau),$$

which need not be invertible. We now write down more details. We begin with the easier operation $\text{\textvisiblespace} \,\square\, G_2$.

Let $G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ and $F_1, F_2 \colon \mathcal{D} \rightrightarrows \mathcal{E}$ be morphisms and let $\sigma \colon F_1 \Rightarrow F_2$ be a transformation. We are going to define a transformation $\sigma \,\square\, G \colon F_1 * G \Rightarrow F_2 * G$. If $x \in \mathcal{C}^0$, then we let

$$(\sigma \,\square\, G)_x := \sigma_{G(x)} \colon F_1\big(G(x)\big) \to F_2\big(G(x)\big).$$

If $x, y \in \mathcal{C}^0$, $f \in \mathcal{C}(x, y)$, then we let

$$(\sigma \,\square\, G)_f = \sigma_{G(f)} \colon F_2(G(f)) \circ \sigma_{G(x)} \Rightarrow \sigma_{G(x)} \circ F_1(G(f)).$$

LEMMA 4.10.1. *The data above defines a transformation* $\sigma \square G \colon F_1 * G \Rightarrow F_2 * G$.

PROOF. The 2-arrows $\sigma_f$ above are clearly natural for 2-arrows in $\mathcal{C}$. The proof that the diagrams (4.3.4) and (4.3.5) commute for $\sigma \,\square\, G$ uses the corresponding diagrams for $\sigma$, the definition of $F_j * G$, and that the 2-arrows $\sigma_f$ are natural for the 2-arrows $\mu^G_{f,g} \colon G(f) \circ G(g) \Rightarrow G(f \circ g)$ and $\lambda^G_x \colon 1_{G^0(x)} \Rightarrow G(1_x)$. $\square$

Let $\sigma$ and $\sigma'$ be two transformations $F_1 \Rightarrow F_2$ and let $\Gamma \colon \sigma \Rrightarrow \sigma'$ be a modification. Define

$$(\Gamma \,\square\, G)_x := \Gamma_{G(x)} \colon \sigma_{G(x)} = (\sigma \,\square\, G)_x \Rightarrow (\sigma' \,\square\, G)_x = \sigma'_{G(x)}$$

for $x \in \mathcal{C}^0$. This is a modification

$$\Gamma \,\square\, G \colon \sigma \,\square\, G \Rrightarrow \sigma' \,\square\, G.$$

This defines a functor

$$\text{\textvisiblespace} \,\square\, G \colon \operatorname{Mor}(F_1, F_2) \to \operatorname{Mor}(F_1 * G, F_2 * G),$$

where $\operatorname{Mor}(F_1, F_2)$ denotes the the category of transformations $F_1 \Rightarrow F_2$ and modifications between them, which is a category of arrows and 2-arrows in $\operatorname{Hom}(\mathcal{C}, \mathcal{D})$.

PROPOSITION 4.10.2. *The operations* $F \mapsto F * G$ *on morphisms,* $\sigma \mapsto \sigma \,\square\, G$ *on transformations and* $\Gamma \mapsto \Gamma \,\square\, G$ *on modifications are part of a strict homomorphism of bicategories*

$$\text{\textvisiblespace} \,\square\, G \colon \operatorname{Mor}(\mathcal{D}, \mathcal{E}) \to \operatorname{Mor}(\mathcal{C}, \mathcal{E}),$$

*which restricts to a strict homomorphism*

$$\text{\textvisiblespace} \,\square\, G \colon \operatorname{Hom}(\mathcal{D}, \mathcal{E}) \to \operatorname{Hom}(\mathcal{C}, \mathcal{E}).$$

*This construction is functorial in the following sense. First,* $\sqcup \mathbin{\Box} 1_{\mathcal{D}}$ *is the identity homomorphism on* $\operatorname{Mor}(\mathcal{C}, \mathcal{D})$. *Secondly, if* $H \colon \mathcal{E} \to \mathcal{E}_2$ *is a morphism, then*

$$(\sqcup \mathbin{\Box} G) \mathbin{\Box} H = \sqcup \mathbin{\Box} (G * H).$$

*In particular,*

$$(\sigma \mathbin{\Box} G) \mathbin{\Box} H = \sigma \mathbin{\Box} (G * H), \qquad (\Gamma \mathbin{\Box} G) \mathbin{\Box} H = \Gamma \mathbin{\Box} (G * H).$$

Proof. We compute

$$\begin{gathered}
\big((\sigma \circ \sigma') \mathbin{\Box} G\big)_x = (\sigma \circ \sigma')_{G(x)} = \sigma_{G(x)} \circ \sigma'_{G(x)} = \big((\sigma \mathbin{\Box} G) \circ (\sigma' \mathbin{\Box} G)\big)_x, \\
(1_F \mathbin{\Box} G)_x = 1_{F(G(x))} = 1_{F*G}(x), \\
(\sigma \mathbin{\Box} 1_{\mathcal{D}})_x = \sigma_x, \\
((\sigma \mathbin{\Box} G) \mathbin{\Box} H)_x = (\sigma \mathbin{\Box} G)_{H(x)} = \sigma_{G(H(x))} = \sigma_{G*H(x)} = (\sigma \mathbin{\Box} (G * H))_x.
\end{gathered}$$

Similar identities hold for arrows instead of objects, giving equalities of transformations; only $\big((\sigma \circ \sigma') \mathbin{\Box} G\big)_f = \big((\sigma \mathbin{\Box} G) \circ (\sigma' \mathbin{\Box} G)\big)_f$ requires another easy computation. □

Next, let $G_1, G_2 \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be morphisms, let $F \colon \mathcal{D} \rightrightarrows \mathcal{E}$ be a homomorphism, and let $\tau \colon G_1 \Rightarrow G_2$ be a transformation. We are going to define a transformation $F \mathbin{\Box} \tau \colon F * G_1 \Rightarrow F * G_2$. If $x \in \mathcal{C}^0$, then we let

$$(F \mathbin{\Box} \tau)_x := F(\tau_x) \colon F(G_1(x)) \to F(G_2(x)).$$

If $x, y \in \mathcal{C}^0$, $f \in \mathcal{C}(x, y)$, then we let $(F \mathbin{\Box} \tau)_f$ be the product 2-arrow

$$\begin{array}{ccccc}
(F * G_2)(f) \circ (F \mathbin{\Box} \tau)_x & = & F(G_2(f)) \circ F(\tau_x) & \xRightarrow{\mu^F_{G_2(f),\tau_x}} & F(G_2(f) \circ \tau_x) \\
\Big\Downarrow (F\mathbin{\Box}\tau)_f & & & & \Big\Downarrow F(\tau_f) \\
(F \mathbin{\Box} \tau)_y \circ (F * G_1)(f) & = & F(\tau_y) \circ F(G_1(f)) & \xLeftarrow[(\mu^F_{\tau_y, G_1(f)})^{-1}]{} & F(\tau_y \circ G_1(f))
\end{array}$$

These 2-arrows are clearly natural. The naturality of the 2-arrows $\mu^F$ and $\lambda^F$ reduces the coherence diagrams (4.3.4) and (4.3.5) for $F \mathbin{\Box} \tau$ to the corresponding diagrams for $\tau$.

**Write more.**

**4.10.2. Equivalences of bicategories.** The products above allow to define equivalences of bicategories.

Definition 4.10.3. An *equivalence* between two bicategories $\mathcal{C}$ and $\mathcal{D}$ consists of homomorphisms $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ such that $G * F$ is equivalent to the identity homomorphism $1_{\mathcal{C}}$ in $\operatorname{Hom}(\mathcal{C}, \mathcal{C})$ and $F * G$ is equivalent to the identity homomorphism $1_{\mathcal{D}}$ in $\operatorname{Hom}(\mathcal{D}, \mathcal{D})$. In other words, there are strong transformations

$$\sigma \colon G * F \Rightarrow 1_{\mathcal{C}}, \quad \sigma^* \colon 1_{\mathcal{C}} \Rightarrow G * F, \qquad \tau \colon F * G \Rightarrow 1_{\mathcal{D}}, \quad \tau^* \colon 1_{\mathcal{D}} \Rightarrow F * G,$$

and invertible modifications

$$\sigma^* \circ \sigma \Rrightarrow 1_{1_{\mathcal{C}}}, \qquad \sigma \circ \sigma^* \Rrightarrow 1_{G*F}, \qquad \tau^* \circ \tau \Rrightarrow 1_{1_{\mathcal{D}}}, \qquad \tau \circ \tau^* \Rrightarrow 1_{1_{\mathcal{D}}}.$$

A homomorphism $F$ is called an *equivalence* if it is part of an equivalence as above.

Isomorphisms of bicategories are, of course, equivalences as well.

Exercise 4.10.4. *Equivalences between bicategories enjoy the 2-out-of-6 property and the 2-out-of-3 property. In detail, let* $\mathcal{C}_1, \dots, \mathcal{C}_4$ *be bicategories and let* $F \colon \mathcal{C}_3 \to \mathcal{C}_4$, $G \colon \mathcal{C}_2 \to \mathcal{C}_3$ *and* $H \colon \mathcal{C}_1 \to \mathcal{C}_2$ *be homomorphisms. If* $F * G$ *and* $G * H$ *are equivalences of bicategories, then so are* $F$, $G$, $H$, *and* $F * G * H$. *If two of* $F$, $G$ *and* $F * G$ *are equivalences, so is the third.*

The Axiom of Choice implies that a functor between categories is an equivalence if and only if it is essentially surjective and fully faithful. The analogous result for bicategories is the following criterion for a homomorphism $F$ to be an equivalence:

THEOREM 4.10.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. A homomorphism $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if all the following statements hold:*

(1) *for each $x \in \mathcal{D}^0$, there are $y \in \mathcal{C}^0$ and an equivalence $F^0(y) \simeq x$ in $\mathcal{D}$;*
(2) *for $y_1, y_2 \in \mathcal{C}^0$ and $g \in \mathcal{D}\big(F^0(y_1), F^0(y_2)\big)$, there are $h \in \mathcal{C}(y_1, y_2)$ and an isomorphism of arrows $F(h) \cong g$ in $\mathcal{D}$;*
(3) *for $y_1, y_2 \in \mathcal{C}^0$ and $g_1, g_2 \in \mathcal{C}(y_1, y_2)$, the map $F$ from 2-arrows $g_1 \Rightarrow g_2$ to 2-arrows $F(g_1) \Rightarrow F(g_2)$ is bijective.*

To make the proof of this theorem more transparent, we first define *inner endomorphisms* of bicategories. Let $\mathcal{C}$ be a bicategory. For each $x \in \mathcal{C}^0$, choose $A^0(x) \in \mathcal{C}^0$ and an equivalence $\alpha_x\colon x \to A^0(x)$. This is part of an adjoint equivalence given by arrows $\alpha_x^*\colon A^0(x) \to x$ and invertible 2-arrows $\upsilon_x\colon 1_x \Rightarrow \alpha_x^* \circ \alpha_x$ and $\varepsilon_x\colon \alpha_x \circ \alpha_x^* \Rightarrow 1_{A^0(x)}$ as in Exercise 4.2.9. We are going to define a homomorphism $A\colon \mathcal{C} \to \mathcal{C}$ and a strong transformation $\alpha\colon 1_{\mathcal{C}} \Rightarrow \mathcal{C}$. The strong transformation $\alpha$ is an equivalence, that is, it is invertible up to modifications. Thus $A$ is equivalent to the identity homomorphism. This makes it an equivalence of bicategories.

As our notation suggests, the homomorphism $A$ acts by the given map $A^0$ on objects. We define

$$A\colon \mathcal{C}(x,y) \to \mathcal{C}\big(A^0(x), A^0(y)\big), \qquad f \mapsto (\alpha_y \circ f) \circ \alpha_x^*, \quad \sigma \mapsto (1_{\alpha_y} \bullet \sigma) \bullet 1_{\alpha_x^*}.$$

This is indeed a functor for the vertical product of 2-arrows. For composable arrows $f \in \mathcal{C}(y,z)$, $g \in \mathcal{C}(x,y)$, define the 2-arrow $\mu^A_{f,g}\colon A(f) \circ A(g) \Rightarrow A(f \circ g)$ as the vertical product

$$\begin{aligned} A(f) \circ A(g) = ((\alpha_z \circ f) \circ \alpha_y^*) \circ ((\alpha_y \circ g) \circ \alpha_x^*) &\cong (\alpha_z \circ f) \circ (\alpha_y^* \circ ((\alpha_y \circ g) \circ \alpha_x^*)) \\ \cong (\alpha_z \circ f) \circ (\alpha_y^* \circ (\alpha_y \circ (g \circ \alpha_x^*))) &\cong (\alpha_z \circ f) \circ ((\alpha_y^* \circ \alpha_y) \circ (g \circ \alpha_x^*)) \\ \xRightarrow{\varepsilon_y} (\alpha_z \circ f) \circ (1_y \circ (g \circ \alpha_x^*)) &\cong (\alpha_z \circ f) \circ (g \circ \alpha_x^*) \\ \cong ((\alpha_z \circ f) \circ g) \circ \alpha_x^* &\cong (\alpha_z \circ (f \circ g)) \circ \alpha_x^* = A(f \circ g). \end{aligned}$$

Here each unlabelled 2-arrow is a horizontal product of unit 2-arrows with a uniter or an associator. For $x \in \mathcal{C}^0$, let $\lambda_x\colon 1_{A^0(x)} \Rightarrow A(1_x)$ be the vertical product

$$1_{A^0(x)} \xRightarrow{\varepsilon_x^{-1}} \alpha_x \circ \alpha_x^* \cong (\alpha_x \circ 1_x) \circ \alpha_x^* = A(1_x).$$

We are given arrows $\alpha_x\colon x \to A^0(x)$ and $\alpha_x^*\colon A^0(x) \to x$ for all $x \in \mathcal{C}^0$. If $x, y \in \mathcal{C}^0$, $f \in \mathcal{C}(x,y)$, we a define a 2-arrow $\alpha_f\colon A(f) \circ \alpha_x \Rightarrow \alpha_y \circ f$ as the vertical product

$$A(f) \circ \alpha_x = ((\alpha_y \circ f) \circ \alpha_x^*) \circ \alpha_x \cong (\alpha_y \circ f) \circ (\alpha_x^* \circ \alpha_x) \xRightarrow{\varepsilon_x} (\alpha_y \circ f) \circ 1_x \cong \alpha_y \circ f$$

and $\alpha_f^*\colon f \circ \alpha_x \Rightarrow \alpha_y^* \circ A(f)$ as the vertical product

$$\begin{aligned} f \circ \alpha_y \cong 1_y \circ (f \circ \alpha_x) \xRightarrow{\upsilon_y} (\alpha_y^* \circ \alpha_y) \circ (f \circ \alpha_x) &\cong \alpha_y^* \circ (\alpha_y \circ (f \circ \alpha_x)) \\ &\cong \alpha_y^* \circ ((\alpha_y \circ f) \circ \alpha_x) = \alpha_y^* \circ A(f). \end{aligned}$$

LEMMA 4.10.6. *The data above defines a homomorphism $A\colon \mathcal{C} \to \mathcal{C}$ and strong transformations $\alpha\colon 1_{\mathcal{C}} \Rightarrow A$ and $\alpha^*\colon A \Rightarrow 1_{\mathcal{C}}$. And $(\upsilon_x)_{x\in\mathcal{C}^0}$ and $(\varepsilon_x)_{x\in\mathcal{C}^0}$ are invertible modifications $1_{1_{\mathcal{C}}} \Rrightarrow \alpha^* \circ \alpha$ and $\alpha \circ \alpha^* \Rrightarrow 1_A$, respectively.*

PROOF. It is obvious that the 2-arrows $\mu_{f,g}$, $\lambda_x$, $\alpha_f$ and $\alpha_f^*$ above are natural for 2-arrows. If $\mathcal{C}$ is strict, then most of the factors in the vertical products above become identities, and it becomes easy to check that $A$ is a homomorphism, $\alpha$ and $\alpha^*$

are strong transformations, and $\upsilon$ and $\varepsilon$ are modifications. In general, this would follow from the Coherence Theorem, and this particular instance of the Coherence Theorem can be checked by hand. □

An equivalence of bicategories $\mathcal{D} \cong \mathcal{E}$ implies equivalences of bicategories $\mathcal{D}^{\mathcal{C}} \cong \mathcal{E}^{\mathcal{C}}$ and $\mathcal{C}^{\mathcal{D}} \cong \mathcal{C}^{\mathcal{E}}$ for all bicategories $\mathcal{C}$. We have already defined equivalences of crossed modules and shown that an equivalence $\mathcal{C} \simeq \mathcal{D}$ induces an equivalence $\mathcal{C}^*(2)^{\mathcal{C}} \cong \mathcal{C}^*(2)^{\mathcal{D}}$ in Theorem 2.8.2. This theorem is a special case of a general fact about bicategories.

Theorem 4.10.7. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be bicategories and let $F\colon \mathcal{D} \to \mathcal{E}$ be an equivalence of bicategories. Then the homomorphisms $\mathcal{D}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{C}}$ and $\mathcal{C}^{\mathcal{D}} \to \mathcal{C}^{\mathcal{E}}$ induced by $F$ are equivalences of bicategories as well.*

The following corollary partly justifies the definition of elementary equivalence for crossed modules and Theorem 2.8.2.

Corollary 4.10.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be crossed modules and let $f\colon \mathcal{C} \to \mathcal{D}$ be an elementary equivalence. Turn $\mathcal{C}$ and $\mathcal{D}$ into 2-categories and view these as bicategories. Then $f$ is an equivalence of bicategories. And the induced strict homomorphism $f^*\colon \mathcal{C}^*(2)^{\mathcal{D}} \to \mathcal{C}^*(2)^{\mathcal{C}}$ is an equivalence of bicategories as well.*

To fully justify the definition of equivalence of crossed modules, we must also prove that two crossed modules that are equivalent as bicategories are equivalent as crossed modules. That is, we may replace a homomorphism that is not strict by a zigzag of strict homomorphisms. This follows from the Yoneda Embedding Theorem for bicategories.

**4.10.3. Leftovers to be done differently.** Let $x, y, z \in \mathcal{C}^0$ and let $f \in \mathcal{C}(x, y)$ be an equivalence, so that $x$ and $y$ are equivalent. Then the functor

$$\mathbb{Y}(f)_z\colon \mathcal{C}(x,z) \to \mathcal{C}(y,z), \qquad g \mapsto g \circ f, \quad \alpha \mapsto \alpha \bullet 1_f,$$

is an equivalence of categories by Corollary 4.7.3. In particular, this functor is fully faithful. The following lemma expands this statement:

Lemma 4.10.9. *Let $\mathcal{C}$ be a bicategory, let $x, y, z \in \mathcal{C}^0$, and let $f \in \mathcal{C}(x, y)$ be an equivalence. Let $g, h \in \mathcal{C}(z, x)$. Then the map from 2-arrows $g \Rightarrow h$ to 2-arrows $f \circ g \Rightarrow f \circ h$ that maps $\alpha\colon g \Rightarrow h$ to $1_f \bullet \alpha$ is bijective. Dually, if $g, h \in \mathcal{C}(y, z)$, then the map from 2-arrows $g \Rightarrow h$ to 2-arrows $g \circ f \Rightarrow h \circ f$ that maps $\alpha\colon g \Rightarrow h$ to $\alpha \bullet 1_f$ is bijective.*

The bicategory structure on $\mathrm{Mor}(\mathcal{C}, \mathcal{D})$ allows us to ask whether a modification is invertible and whether a transformation is an equivalence (see Definition 4.2.5). It is rather easy to characterise the invertible modifications, so that we leave this as an exercise. Then we characterise which transformations are equivalences.

Exercise 4.10.10. *A modification $\Gamma = (\Gamma_x)_{x\in\mathcal{C}^0}$ is invertible in the bicategory $\mathrm{Mor}(\mathcal{C}, \mathcal{D})$ if and only if each 2-arrow $\Gamma_x$ in it is invertible. The inverse modification is $\Gamma^{-1} = (\Gamma_x^{-1})_{x\in\mathcal{C}^0}$.*

Theorem 4.10.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories, let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be morphisms. Let $\sigma\colon F \Rightarrow G$ be a transformation, consisting of arrows $\sigma_x\colon F^0(x) \to G^0(x)$ for $x \in \mathcal{C}^0$ and 2-arrows $\sigma_f\colon G(f) \circ \sigma_x \Rightarrow \sigma_y \circ F(f)$ for $f \in \mathcal{C}(x, y)$, $x, y \in \mathcal{C}^0$. The transformation $\sigma_x$ is an equivalence if and only if it is strong and each $\sigma_x$ is an equivalence in $\mathcal{D}$.*

Proof. Assume first that $\sigma$ is an equivalence. Then there is a transformation $\tau\colon G \Rightarrow F$ such that the transformations $\tau \circ \sigma$ and $\sigma \circ \tau$ are equivalent to the unit transformations on $F$ and $G$, respectively. Then there are invertible 2-arrows $\tau_x \circ \sigma_x \Rightarrow 1_{F^0(x)}$ and $\sigma_x \circ \tau_x \Rightarrow 1_{G^0(x)}$ for all $x \in \mathcal{C}^0$. This says that each $\sigma_x$ is an equivalence. We claim that $(\tau \circ \sigma)_f$ is an invertible 2-arrow. This is true because the 2-arrows $(1_F)_f$ in the unit transformation are invertible, being made out of uniters, and the 2-arrows $(\tau \circ \sigma)_f$ and $(1_F)_f$ are intertwined by invertible 2-arrows from the invertible modification $\tau_x \circ \sigma_x \Rightarrow 1_{F^0(x)}$. Since $(\tau \circ \sigma)_f$ is invertible, the definition of $(\tau \circ \sigma)_f$ shows that $1_{\tau_x} \bullet \sigma_f$ is right invertible. Since $\tau_x$ is an equivalence, Lemma 4.10.9 shows that the map $1_{\tau_x} \bullet \sqcup$ is bijective. It is a homomorphism for the vertical product as well. Therefore, it preserves and detects right invertibility. So $\sigma_f$ is right invertible. A dual argument shows that $\sigma_f$ is left invertible. So $\sigma_f$ is invertible, that is, the transformation is strong.

Conversely, assume that each $\sigma_x$ is an equivalence and that each $\sigma_f$ is invertible. Let $\tau_x \colon G^0(x) \to F^0(x)$ be an equivalence that is inverse to $\sigma_x$ up to invertible 2-arrows $\alpha_x \colon \tau_x \circ \sigma_x \Rightarrow 1_{F^0(x)}$ and $\beta_x \colon \sigma_x \circ \tau_x \Rightarrow 1_{G^0(x)}$. We ask these to form an adjoint equivalence as in Exercise 4.2.9. We are going to define natural 2-arrows $\tau_f \colon F(f) \circ \tau_x \Rightarrow \tau_y \circ G(f)$ for all $f \in \mathcal{C}(x, y)$ that make $(\tau_x, \tau_f)$ into a transformation $\tau \colon G \Rightarrow F$ and $(\alpha_x)$ and $(\beta_x)$ into modifications $\tau \circ \sigma \Rrightarrow 1_F$ and $\sigma \circ \tau \Rrightarrow 1_G$. These modifications are invertible by Exercise 4.10.10. So the claims above say that $\sigma$ is an equivalence. We define $\tau_f$ as the vertical product

$$\begin{aligned} F(f) \circ \tau_x &\cong (1_{F^0(y)} \circ F(f)) \circ \tau_x \xrightarrow{\alpha_y^{-1}} ((\tau_y \circ \sigma_y) \circ F(f)) \circ \tau_x \\ &\cong (\tau_y \circ (\sigma_y \circ F(f))) \circ \tau_x \xrightarrow{(1 \bullet \sigma_f^{-1}) \bullet 1} (\tau_y \circ (G(f) \circ \sigma_x)) \circ \tau_x \\ &\cong \tau_y \circ (G(f) \circ (\sigma_x \circ \tau_x)) \xrightarrow{1 \bullet (1 \bullet \beta_x)} \tau_y \circ (G(f) \circ 1_{G^0(x)}) \cong \tau_y \circ G(f). \end{aligned}$$

Here the unlabelled 2-arrows are built from uniters and associators, and $\sigma_f^{-1}$ exists because the transformation $\sigma$ is strong. We claim that the arrows $\tau_x$ for $x \in \mathcal{C}^0$ and the 2-arrows $\tau_f$ for $f \in \mathcal{C}$ form a transformation $\tau$. The naturality of $\tau_f$ follows easily from the naturality of associators, uniters and of $\sigma_f$. To prove the commuting diagram (4.3.4) for $\tau$, it suffices by Lemma 4.10.9 if it commutes after taking horizontal products with $1_{\sigma_z}$ on the left and with $1_{\sigma_x}$ on the right. And this follows with some computation from the corresponding diagram for the transformation $\sigma$. The only complication is to keep track of the various associators and uniters. And the Coherence Theorem (Theorem 4.8.2) implies that parallel 2-arrows built in this way commute automatically. The same trick also works for the commuting diagram (4.3.5). So $\tau$ is a transformation.

We claim that $(\alpha_x)$ is a modification $\tau \circ \sigma \Rrightarrow 1_F$. To check this, we must prove that certain 2-arrows $F(f) \circ \tau_x \circ \sigma_x \Rightarrow \tau_x \circ \sigma_x \circ F(f)$ are equal. Again by Lemma 4.10.9, it suffices to prove that their horizontal product on the left with $1_{\sigma_y}$ are equal. This allows us to use the definition of $\tau_f$ to check the equality. The same trick shows that $(\beta_x)$ is a modification $\sigma \circ \tau \Rrightarrow 1_G$. □

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