# Exercise sheet 1.

Name

Exercise group (tutor's name)

| **Exercise** | **1** | **2** | **3** | **4** | **Σ** |
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| **Points** | | | | | |

Deadline: **Friday, 29.10.2021, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

**Exercise 1.** Let $(P, \le)$ be a preorder, that is, $\le$ is a reflexive and transitive relation. Construct a category with object set $P$ and with exactly one arrow $x \to y$ if $x \le y$, and no arrow $x \to y$ otherwise; there is exactly one such category, called the *category associated to the preorder*. Conversely, if $\mathcal{C}$ is a category, define a relation on its object set by $x \le y$ if there is an arrow $x \to y$. Show that this relation is reflexive and transitive, hence a preorder. Show that $\mathcal{C}$ is isomorphic to the category associated to this preorder if and only if for all objects $x$ and $y$ of $\mathcal{C}$, there is at most one arrow $x \to y$.

**Exercise 2.** Let $f\colon x \to y$ and $g, h\colon y \rightrightarrows x$ be arrows in a category with $f \circ g = \mathrm{id}_y$ and $h \circ f = \mathrm{id}_x$. Show that $g = h$; so $f$ is an isomorphism. Conclude that an arrow has at most one inverse.

**Exercise 3.** Let $\mathcal{C}$ be a category. Let $\mathcal{C}_{\mathrm{epi}}$, $\mathcal{C}_{\mathrm{mono}}$ and $\mathcal{C}^*$ denote the epimorphisms, the monomorphisms, and the isomorphisms in $\mathcal{C}$, respectively. Show that these are closed under composition and contain the identities. That is, they form subcategories with the same objects as $\mathcal{C}$.

Also prove the following. Let $f\colon x \to y$ and $g\colon y \to z$ be morphisms. If $gf$ is monic, then $f$ is monic. If $gf$ is epic, then $g$ is epic.

**Exercise 4.** Let $\mathfrak{Nor}$ be the category of normed spaces with bounded linear maps as morphisms. Let $f\colon X \to Y$ be a morphism in $\mathfrak{Nor}$. Show that $f$ is a monomorphism if and only if it is injective, and an epimorphism if and only if its range is dense. You may use the following without proof: if $Z \subseteq Y$ is a closed subspace, then the quotient space $Y/Z$ with the quotient norm is again a normed space. Find a morphism $f$ in $\mathfrak{Nor}$ that is both monic and epic, but not an isomorphism.