Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Devarshi Mukherjee.

Exercise sheet 2.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Friday, 5.11.2021, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

The following exercises are about *functors*. The relevant reading material is Section 1.3 from Emily Riehl's book *Category Theory in Context*.

Exercise 1. Find an example of a functor $F\colon C \to D$ for which the objects and morphisms in the image do not define a subcategory of D.

Exercise 2. Let $F\colon D \to C$ and $G\colon E \to C$ be functors. Show that there is a category – called the *comma category* $F \downarrow G$ – which has

- as objects, triples $(d \in D, e \in E, f\colon Fd \to Ge \in C)$;
- as morphisms $(d, e, f) \to (d', e', f')$, a pair of morphisms $h\colon d \to d'$ and $k\colon e \to e'$ such that the following square commutes:

$$\begin{array}{ccc} Fd & \xrightarrow{f} & Ge \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle Gk} \\ Fd' & \xrightarrow{f'} & Ge' \end{array}$$

Exercise 3. What is a functor between two groups, viewed as categories?

What is a functor between two preorders, viewed as categories?

Exercise 4. Functoriality sometimes depends on the categories that are chosen. Consider three standard constructions from group theory. Let G be a group. Its *centre* is

$$Z(G) = \{h \in G : hg = gh \text{ for all } g \in G\}.$$

Its *commutator subgroup* $[G, G] \subseteq G$ is the subgroup generated by all "commutators" $ghg^{-1}h^{-1}$, $g, h \in G$. (This subgroup is normal, and is the smallest normal subgroup N so that G/N is Abelian.) Its automorphism group $\mathrm{Aut}(G)$ is the group of all group isomorphisms from G to itself. Let

$$\mathfrak{Groups} \supseteq \mathfrak{Groups}_{\mathrm{epi}} \supseteq \mathfrak{Groups}^*$$

be the categories of groups with *all* group homomorphisms, *surjective* group homomorphisms, and *bijective* group homomorphisms as arrows. For each of the three constructions $Z(G)$, $[G, G]$, $\mathrm{Aut}(G)$, does it come from a functor $\mathfrak{Groups} \to \mathfrak{Groups}$, $\mathfrak{Groups}_{\mathrm{epi}} \to \mathfrak{Groups}$, or $\mathfrak{Groups}^* \to \mathfrak{Groups}$?

(A group homomorphism is an epimorphism in $\mathfrak{Groups}$ if and only if it is surjective. The proof of this theorem is due to Otto Schthere exists no homomorphism from Z toreier and uses amalgamated free products of groups. You need not prove this!)