Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Devarshi Mukherjee.

Exercise sheet 3.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Friday, 12.11.2021, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

The following exercises are about naturality and natural transformations. The relevant reading material is Section 1.4 from Emily Riehl's book *Category Theory in Context*.

Exercise 1. What is a natural transformation between a parallel pair of functors between groups, regarded as one-object categories?

What is a natural transformation between a parallel pair of functors between preorders, regarded as categories?

Exercise 2. Fix a field K. A *bilinear form* is a finite-dimensional K-vector space V with a nondegenerate bilinear map $V \times V \to K$; nondegeneracy means that $f(v,w) = 0$ for all $v \in V$ implies $w = 0$ and $f(v,w) = 0$ for all $w \in V$ implies $v = 0$. Let (V,b) and (W,c) be bilinear forms. A map $f\colon (V,b) \to (W,c)$ is a linear map $V \to W$ with $c(f(x), f(y)) = b(x,y)$ for all $x, y \in V$. Let $\mathcal{B}$ be the category with bilinear forms as objects, the above maps as arrows, and the obvious product and units. For a bilinear form (V,b), define a linear map $V \to V^*$ by mapping $v \in V$ to the functional $V \to K$, $w \mapsto b(v,w)$. This map is injective because b is nondegenerate and hence an isomorphism.

Lift the map $(V,b) \mapsto V^*$ to a *covariant* functor $\mathcal{B} \to \mathcal{B}$, in such a way that the isomorphisms $V \cong V^*$ described above become a natural isomorphism $(V,b) \cong (V^*, b^*)$ between the identity functor and the "covariant dual space functor" on $\mathcal{B}$. Thus you have to describe a canonical bilinear form b^* on V^* and how a map $(V,b) \to (W,c)$ induces a map $(V^*, b^*) \to (W^*, c^*)$, in a functorial way, such that the isomorphisms $(V,b) \cong (V^*, b^*)$ are natural. Is there more than one way to do this?

Exercise 3. Describe the centres of the categories of sets and of R-modules for a given ring R. Recall that the centre is the set of all natural transformations from the identity functor to itself. (For this exercise, the centre is just a set.)

Exercise 4. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. Show that the functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between such functors are the objects and arrows of a category $\mathcal{D}^{\mathcal{C}}$. That is, describe a natural way to compose natural transformations $F \Rightarrow G \Rightarrow H$ between three functors F, G, H from $\mathcal{C}$ to $\mathcal{D}$ and show that your composition is associative and describe its units. Show that the isomorphisms in this category are the natural isomorphisms, and describe the inverse natural isomorphism $\Phi^{-1}\colon G \Rightarrow F$ of a natural isomorphism $\Phi\colon F \Rightarrow G$.