Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Devarshi Mukherjee.

Categories in context
Winter semester 2021–2

Exercise sheet 7.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Friday, 10.12.2021, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

The following exercises are from Section 3.3 of Emily Riehl's book *Categories in context.*

Exercise 1. Show that an equivalence of categories $F\colon \mathcal{C} \to \mathcal{D}$ preserves all limits and colimits that exist in $\mathcal{C}$ and creates all limits and colimits that $\mathcal{D}$ admits.

Exercise 2. Let $\mathcal{C}, \mathcal{D}, \mathcal{J}$ be categories, with $\mathcal{J}$ small. Let $K\colon \mathcal{J} \to \mathcal{C}$ be a diagram in $\mathcal{C}$ of shape $\mathcal{J}$ and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Define a canonical map $\operatorname{colim} FK \to F(\operatorname{colim} K)$. Show that the functor F preserves colimits if and only if this canonical map is an isomorphism.

Exercise 3. Let $\mathfrak{Mod}_R$ and $\mathfrak{Mod}_S$ be the categories of modules over two rings R and S and let $F\colon \mathfrak{Mod}_R \to \mathfrak{Mod}_S$ be a functor that is additive in the sense that $F(f+g) = F(f) + F(g)$ if $f, g\colon M \rightrightarrows N$ are two parallel module homomorphisms. Prove that F preserves colimits if and only if F preserves coproducts and cokernels. Preserving cokernels means that for any $\varphi\colon M \to N$, the canonical map $F(\operatorname{coker}(\varphi)) \to \operatorname{coker} F(\varphi)$ is an isomorphism.

Prove that the second property is equivalent to F being "right exact": whenever $M' \subseteq M$ is a submodule and $M'' = M/M'$, then $F(M') \to F(M) \to F(M'') \to 0$ is "exact," that is, the map $F(M) \to F(M'')$ (induced by the quotient map $M \to M''$) is surjective and its kernel is the image of the map $F(M') \to F(M)$ induced by the inclusion map $M' \to M$.

Exercise 4. Let $\mathfrak{Mod}_R$ and $\mathfrak{Mod}_S$ be the categories of modules over two rings R and S and let B be an R,S-bimodule. Let $\operatorname{Hom}_R(B, M)$ be the set of R-module homomorphisms $B \to M$, with its obvious abelian group structure. Show that $(sf)(b) = f(bs)$ for $s \in S$, $b \in B$, $f \in \operatorname{Hom}_R(B, M)$ defines a natural S-module structure on $\operatorname{Hom}_R(B, M)$, so that $\operatorname{Hom}_R(B, -)$ becomes a functor $\mathfrak{Mod}_R \to \mathfrak{Mod}_S$. Show that this functor preserves limits.