Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Devarshi Mukherjee.

Exercise sheet 9.

Name

Exercise group (tutor's name)

Exercise	**1**	**2**	**3**	**4**	**Σ**
Points					

Deadline: **Friday, 14.1.2021, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. Show that a collection of isomorphisms $\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd)$, $f^\# \mapsto f^\flat$, for all $c \in \mathcal{C}^0$, $d \in \mathcal{D}^0$ is natural – and hence an adjunction – if and only if for any objects $c, c' \in \mathcal{C}^0$, $d, d' \in \mathcal{D}^0$ and any morphisms $f^\#\colon Fc \to d$, $g^\#\colon Fc' \to d'$, $h\colon c \to c'$, $k\colon d \to d'$, the left-hand square below commutes in $\mathcal{D}$ if and only if the right-hand transposed square below commutes in $\mathcal{C}$:

$$\begin{array}{ccc} Fc & \xrightarrow{f^\#} & d \\ \downarrow{\scriptstyle Fh} & & \downarrow{\scriptstyle k} \\ Fc' & \xrightarrow{g^\#} & d' \end{array} \qquad \begin{array}{ccc} c & \xrightarrow{f^\flat} & Gd \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Gk} \\ c' & \xrightarrow{g^\flat} & Gd' \end{array}$$

Exercise 2. Construct left and right adjoint functors for the functor $\mathfrak{Cat} \to \mathfrak{Set}$ that maps a category to its set of objects, and prove the adjointness.

Exercise 3. Let R be a ring and let $U\colon \mathfrak{Mod}_R \to \mathfrak{Ab}$ be the forgetful functor to Abelian groups. Show that the functor $R \otimes_{\mathbb{Z}} -\colon \mathfrak{Ab} \to \mathfrak{Mod}_R$ is adjoint to U by describing the unit and counit and checking the triangle identities.

Exercise 4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let F and G be adjoint functors, with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$. Let $\mathcal{C}_1 \subseteq \mathcal{C}$ be the full subcategory whose objects are all $c \in \mathcal{C}^0$ for which η_c is an isomorphism, and let $\mathcal{D}_1 \subseteq \mathcal{D}$ be the full subcategory whose objects are all $d \in \mathcal{D}^0$ for which ε_d is an isomorphism. Show that $F(\mathcal{C}_1) \subseteq \mathcal{D}_1$ and $G(\mathcal{D}_1) \subseteq \mathcal{C}_1$ and that these restrictions $\mathcal{C}_1 \leftrightarrow \mathcal{D}_1$ are equivalences of categories that are inverse to each other.