Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Devarshi Mukherjee.

Exercise sheet 10.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Friday, 21.1.2021, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Assume that $F\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$ with counit $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ and unit $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$. Show that

$$\mathcal{D}(d_1, d_2) \xrightarrow{\varepsilon^*} \mathcal{D}(FGd_1, d_2) \xrightarrow[\cong]{\text{adjunction}} \mathcal{C}(Gd_1, Gd_2)$$

is the functor G. Conclude that

1. G is faithful if and only if each arrow $\varepsilon_d\colon FGd \to d$ is an epimorphism;
2. G is full if and only if each arrow $\varepsilon_d\colon FGd \to d$ is a split monomorphism;
3. G is full and faithful if and only if ε is an isomorphism.

Exercise 2. Let $\mathcal{C}$ be a category and let $\mathbb{1}$ be the one-object one-arrow category. When does the constant functor $\mathcal{C} \to \mathbb{1}$ have a left adjoint, and what is this left adjoint functor? When does it have a right adjoint, and what is the right adjoint functor?

Exercise 3. Let $\mathcal{C}$ be a locally small category and let $\mathcal{J}$ be a small category. Assume that $\mathcal{C}$ has all $\mathcal{J}$-shaped limits and colimits. Describe the unit and counit for the adjunctions between the constant diagram functor $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ and the limit and colimit functors $\mathcal{C}^{\mathcal{J}} \rightrightarrows \mathcal{C}$.

Exercise 4. Show that the functor from sets to topological spaces that equips a set with the discrete topology commutes with arbitrary colimits and with finite limits, but not with infinite products. Conclude that this functor does not have a left adjoint.