Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Devarshi Mukherjee.

Exercise sheet 12.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Friday, 4.2.2022, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let k be a vector space, let $\mathfrak{Vect}_k$ be the category of k-vector spaces and let $\mathfrak{Alg}_k$ be the category of k-algebras. The forgetful functor $\mathfrak{Alg}_k \to \mathfrak{Vect}_k$ has a left adjoint, mapping a k-vector space V to the free k-algebra TV on V, which is also called the *tensor algebra of* V because its underlying k-vector space is $TV = \bigoplus_{n=0}^{\infty} V^{\otimes_k n}$.

1. The adjunction above yields a monad (T, η, μ). Describe its unit and multiplication.
2. Prove that an algebra over this monad is equivalent to a k-algebra and that morphisms of T-algebras are the same as k-algebra homomorphisms.

Exercise 2. The ultrafilters on a set S form a subset of the double powerset $\mathcal{P}^2(S)$. The latter is a monad in a canonical way, compare the construction for $\mathcal{P}(S)$. Describe the unit and the multiplication for the monad $\mathcal{P}^2$, and show that it restricts to the subset of ultrafilters, which therefore gives a "submonad" and becomes a monad in its own right. (These are the unit and multiplication from the adjunction involving the forgetful functor from compact Hausdorff spaces to sets and the Stone–Čech compactification.)

Exercise 3. Let (T, η, μ) be a monad on a category $\mathcal{C}$. Show that the following are equivalent:

1. $\mu\colon T^2 \Rightarrow T$ is a natural isomorphism;
2. the natural transformations $\eta T, T\eta\colon T \Rightarrow T^2$ are equal;
3. for each $A \in \mathcal{C}^0$, the map $\mu_A\colon T^2A \to TA$ is a monomorphism.

The first property is the reason why such monads are called *idempotent*. For instance, the adjunction for a reflective subcategory $\mathcal{D} \subseteq \mathcal{C}$ gives an idempotent monad.

Exercise 4. Let (T, η, μ) be an idempotent monad on a category $\mathcal{C}$, that is, $\mu\colon T^2 \Rightarrow T$ is a natural isomorphism. Show that the category of T-algebras is equivalent to a reflective subcategory of $\mathcal{C}$.

Thus idempotent monads are "equivalent" to reflective subcategories.