Mathematisches Institut
Georg-August-Universität Göttingen
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Noncommutative geometry I
Summer semester 2024

Exercise sheet 6.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	5	6	Σ
Points							

Deadline: **Monday, 20.5.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise gives a nice application of Kadison's Transitivity Theorem.) Let A be a C^*-algebra and let $\pi\colon A \to \mathbb{B}(\mathcal{H})$ be an irreducible representation on a Hilbert space $\mathcal{H}$. Assume that $\pi(A)$ contains at least one nonzero compact operator. Show that $\pi(A)$ contains all operators of rank one. Deduce further that $\pi(A)$ contains all compact operators on $\mathcal{H}$.

Exercise 2. (This exercise describes the structure of the Toeplitz C^*-algebra, that is, the C^*-algebra generated by the unilateral shift operator. More precisely, we describe it as an extension of the C^*-algebra of compact operators by the commutative C^*-algebra of continuous functions on the unit circle. This result will turn out to be crucial (in the next semester) for certain K-theory computations.) Let S be the unilateral shift operator on $\ell^2(\mathbb{N})$ defined by $S(\delta_n) := \delta_{n+1}$.

1. Show that the commutant of $\mathrm{C}^*(S)$ is $\mathbb{C} \cdot 1$ and deduce that the standard representation $\mathrm{C}^*(S) \to \mathbb{B}(\ell^2\mathbb{N})$ is irreducible. Use the previous exercise and that $1 - SS^*$ is compact to show that $\mathbb{K}(\ell^2(\mathbb{N})) \subseteq \mathrm{C}^*(S)$.
2. Show that the quotient $\mathrm{C}^*(S)/\mathbb{K}(\ell^2(\mathbb{N}))$ is a commutative C^*-algebra generated by $\tilde{S}$, the image of S in $\mathrm{C}^*(S)/\mathbb{K}(\ell^2(\mathbb{N}))$ under the canonical projection. Deduce that $\tilde{S}$ is a unitary operator.
3. Compute the spectrum of $\tilde{S}$. (Hint: Use Gelfand Naimark to find a closed, non-empty subset $X \subseteq \mathbb{T}$ such that $\mathrm{C}^*(\tilde{S}) \cong C(X)$ and then show that we have $X = \mathbb{T}$).

Exercise 3. (This exercises clarifies the close link between hereditary C^*-subalgebras and closed left ideals. The latter are, of course, equivalent to closed right ideals by taking adjoints.) Let A be a C^*-algebra and let L be a closed left ideal of A. Show that $L \cap L^*$ is a hereditary C^*-subalgebra of A. Show that the map $L \mapsto L \cap L^*$ is a bijection from the set of closed left ideals of A to the set of hereditary subalgebras of A. (Hint: If B is a hereditary subalgebra of A, then the set

$$L_B := \{a \in A : a^*a \in B\}$$

is the unique closed left ideal of A corresponding to B.)

Exercise 4. (This exercise describes the positive linear functionals and states of matrix algebras, which is an elementary example. A state is defined as a positive linear functional of norm 1. For a unital C^*-algebra, it is the same as a positive linear functional l with $l(1) = 1$.)

1. Let $\mathbb{M}_n(\mathbb{C})$ be the C^*-algebra of $n \times n$ matrices over $\mathbb{C}$. Show that every linear functional on $\mathbb{M}_n(\mathbb{C})$ is of the form
$$f_b(a) = \mathrm{tr}(a \cdot b)$$
for some $b \in \mathbb{M}_n(\mathbb{C})$. Show that a linear functional f_b is positive if and only if b is positive. (Hint: Recall that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ and show that $\mathrm{tr}(x) \geq 0$ if $x \geq 0$).
2. Use Lemma 1.9.5 from Davidson's book to deduce that the state space of $\mathbb{M}_n(\mathbb{C})$ is
$$\mathcal{S}(\mathbb{M}_n(\mathbb{C})) = \{b \in \mathbb{M}_n(\mathbb{C}) : b \geq 0, \quad \mathrm{tr}(b) = 1\}.$$

Exercise 5. Show that $\mathbb{M}_n(\mathbb{C})$ has (up to unitary equivalence), a unique irreducible representation on $\mathbb{C}^n$, given by matrix-vector multiplication. Use Theorem 1.9.8 to show that the extreme points of $\mathcal{S}(\mathbb{M}_n(\mathbb{C}))$ are given by $\{|v\rangle\langle v| : v \in \mathbb{C}^n, \quad \|v\| = 1\}$.

Exercise 6. Let μ be a regular, Borel probability measure on a compact space X. Show that

$$\int_X : C(X) \to \mathbb{C}, \quad f \mapsto \int_X f(x) \mathrm{d}\mu(x)$$

is a state and compute its associated GNS representation.