Exercise sheet 7.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Monday, 27.5.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise characterises the nondegenerate representations of a C*-algebra in several equivalent ways.) Let A be a C*-algebra, $\mathcal{H}$ a Hilbert space, and $\pi\colon A \to \mathbb{B}(\mathcal{H})$ a representation. Let $(e_n)_{n\in N}$ be an approximate unit in A. Show that the following three statements are equivalent:

1. $\overline{\{\pi(a)\xi : a \in A,\ \xi \in \mathcal{H}\}} = \mathcal{H}$;
2. (nondegenerate) the linear span of $\{\pi(a)\xi : a \in A,\ \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$;
3. $\lim \pi(e_n) = \mathrm{id}_{\mathcal{H}}$ in the strong topology.
4. (trivial null space) $\{\xi \in \mathcal{H} : \pi(a)\xi = 0 \text{ for all } a \in A\} = \{0\}$.

Deduce that cyclic and irreducible representations satisfy these conditions.

Exercise 2. (This exercise describes the universal C*-algebra generated by a unitary.) Show that the universal C*-algebra generated by a unitary exists and is naturally isomorphic to $\mathrm{C}(\mathbb{T})$.

Exercise 3. (This exercise shows that C*-hulls do not always exist.) Make $\mathbb{C}[x]$ a *-algebra by $x^* := x$. Show that there is no maximal C*-seminorm on this *-algebra, so that the C*-hull does not exist.

Exercise 4. (This exercise describes the positive linear functionals and the pure states on a matrix algebra. As a consequence, it also shows that any irreducible representation of $\mathbb{M}_n(\mathbb{C})$ is unitarily equivalent to the standard representation on $\mathbb{C}^n$.)

1. Show that any linear map $f\colon \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$ is of the form $f(a) := \mathrm{tr}(a \cdot b)$ for a unique $b \in \mathbb{M}_n(\mathbb{C})$. In the following, we fix f and b like this.
2. Show that $f(a^*) = \overline{f(a)}$ for all $a \in \mathbb{M}_n(\mathbb{C})$ if and only if $b = b^*$;
3. Show that f is positive if and only if $b \geq 0$.
4. Show that f is a pure state if and only if b is a rank-one orthogonal projection, that is, $b = |\xi\rangle\langle\xi|$ for some unit vector $\xi \in \mathbb{C}^n$.
5. What is the cyclic representation of $\mathbb{M}_n(\mathbb{C})$ associated to the pure state $a \mapsto \mathrm{tr}\big(a|\xi\rangle\langle\xi|\big)$?