Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Akshara Prasad.

Exercise sheet 8.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Monday, 3.6.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Show that every finite-dimensional C*-algebra may be faithfully represented on a finite-dimensional Hilbert space. (Hint: Show that finitely many states suffice for the Gelfand–Naimark Theorem in this context).

Exercise 2. (This exercise uses the polar decomposition to prove that two irreducible representations are equivalent once there is a nonzero intertwiner between them.) Let π and ρ be representations of a C*-algebra A on Hilbert spaces H_π and H_ρ, respectively. Let $T \in \mathbb{B}(\mathcal{H}_\rho, \mathcal{H}_\pi)$ intertwine π and ρ, that is,

$$T \circ \rho(a) = \pi(a) \circ T$$

for all $a \in A$. Assume $T \neq 0$.

1. Show that the partial isometry U in the polar decomposition of T also intertwines π and ρ.
2. Show that if ρ is irreducible, then ρ is unitarily equivalent to a subrepresentation of π. If π and ρ are both irreducible, then they are unitarily equivalent.

Exercise 3. (The lattice of ideals in a C*-algebra is an obvious and important invariant. This exercise explains how to encode this lattice differently: it is isomorphic to the lattice of open subsets for a canonical topology on the set of irreducible representations of A. Let A be a C*-algebra. Let $\hat{A}$ be the set of unitary equivalence classes of irreducible representations of A. For an ideal $I \subseteq A$, let $\hat{I} \subseteq \hat{A}$ be the subset of all irreducible representations with nonzero restriction to I. The set of ideals in A is partially ordered by the inclusion relation $\subseteq$.

1. Show that $x \in A$ belongs to an ideal I if and only if $\pi(x) = 0$ for all $\pi \in \hat{A} \setminus \hat{I}$. As a consequence, the map $I \mapsto \hat{I}$ is injective.
2. Show that two ideals $I, J \subseteq A$ satisfy $\hat{I} \subsetneq \hat{J}$ if and only if $I \subsetneq J$. Roughly speaking, the map $I \mapsto \hat{I}$ is an order isomorphism from the partially ordered set of ideals in A to some set of subsets of $\hat{A}$.
3. Show that the set of subsets of A of the form $\hat{I}$ for ideals I in A is a topology on $\hat{A}$, that is, it is closed under arbitrary unions and finite intersections.

Exercise 4. Ideals and the space of irreducible representations in the commutative case. Let A be a commutative C*-algebra.

1. Show that any irreducible representation of A has dimension 1 and is given by a character $A \to \mathbb{C}$.
2. Show that any ideal in $\mathrm{C}_0(X)$ for a locally compact space X is of the form $\mathrm{C}_0(U)$ for an open subset $U \subseteq X$. You may use Exercise 3.1.
3. Show that the set of irreducible representations of A with the topology described above is homeomorphic to the topological space of characters of A with the weak topology as in the Gelfand–Naimark Theorem.