Exercise sheet 9.

Name

Exercise group (tutor's name)

Exercise	1	2	3	Σ
Points				

Deadline: **Monday, 10.6.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise relates the set of irreducible representations of a C*-algebra to the character set of its centre. We are going to apply this to noncommutative tori with rational parameter below.) Let $\hat{A}$ denote the set of unitary equivalence classes of irreducible representations of a C*-algebra A. Show that the centre $Z(A)$ of A acts by scalars on any irreducible representation of A. Use this to define a canonical map $q\colon \hat{A} \to \widehat{Z(A)}$.

The following two exercises deal with noncommutative tori. We have defined them as the crossed product $\mathrm{C}^*(\mathrm{C}(\mathbb{T}), \varphi_\lambda^*)$ for the homeomorphism $z \mapsto \lambda z$ of the circle $\mathbb{T}$. Here we use a slightly different, but equivalent definition. Let A_λ^0 be the unital *-algebra generated by two elements U and V with the relations $\lambda UV = VU$ and $UU^* = U^*U = VV^* = V^*V = 1$ (U and V are unitary). Let

$$A_\lambda := \mathrm{C}^*(U, V \mid U, V \text{ unitaries}, \lambda UV = VU)$$

for some $\lambda = \mathrm{e}^{2\pi\mathrm{i}\theta}$, $\theta \in \mathbb{R}$, be its C*-hull, the Hausdorff completion in the maximal C*-seminorm on A_λ^0.

Exercise 2. (In this exercise, we first describe an obvious basis for A_λ^0, then build a canonical cyclic representation and study the corresponding state.)

1. Show $U^n V^m = \lambda^{-nm} V^m U^n$ for $n, m \in \mathbb{Z}$ and deduce that A_λ^0 is spanned by $U^n V^m$ for $n, m \in \mathbb{Z}$.
2. Define $\tilde{U}, \tilde{V} \in \mathbb{B}(\ell^2(\mathbb{Z}^2))$ by $\tilde{U} f(n,m) = f(n-1, m)$ and $\tilde{V} f(n,m) = \lambda^n f(n, m-1)$. Show that they satisfy the relations of A_λ, so that they generate a representation π of A_λ on $\mathbb{B}(\ell^2(\mathbb{Z}^2))$.
3. Show that $\{\tilde{U}^n \tilde{V}^m : n, m \in \mathbb{Z}\}$ are linearly independent. Deduce that $\{U^n V^m : n, m \in \mathbb{Z}\}$ is a basis for A_λ^0 and that the canonical map from A_λ^0 to A_λ is injective.
4. Show that $\delta_{0,0}$ is a cyclic vector for the representation π. Show that the corresponding state on A_λ is the state $\tau\colon A_\lambda \to \mathbb{C}$, $\tau(U^m V^n) := \delta_{m,0}\delta_{n,0}$. Deduce that $\|\tau\| = 1$. Prove that $\tau(ab) = \tau(ba)$ holds for all $a, b \in A_\lambda$. Such states are called *tracial*.

Exercise 3. Now let $\lambda = \mathrm{e}^{2\pi\mathrm{i}n/m}$ with coprime $n, m \in \mathbb{N}_{\geq 1}$. So $\lambda^m = 1$ and $\lambda^k \neq 1$ for $1 \leq k < m$.

1. Show that V^m and U^m lie in the centre $Z(A_\lambda)$ of A_λ. Show that the C*-subalgebra generated by them is isomorphic to $\mathrm{C}(\mathbb{T}^2)$.
2. Let $(z, w) \in \mathbb{T}^2$. Define the *fibre* $A_\lambda(z,w)$ *of* A_λ *at* (z,w) by
$$A_\lambda(z,w) := A_\lambda \,/\, \overline{\mathrm{C}_0(\mathbb{T}^2 \setminus \{(z,w)\}) \cdot A_\lambda}, \qquad \mathrm{C}_0(\mathbb{T}^2 \setminus \{(z,w)\}) = \{f \in \mathrm{C}(\mathbb{T}^2) : f(z,w) = 0\}.$$
Let $(e_j)_j$ be the standard basis vectors of $\mathbb{C}^n$ and define
$$\hat{U} e_j := \begin{cases} e_{j+1} & \text{if } 1 \leq j \leq m, \\ w e_1 & \text{if } j = m, \end{cases} \qquad \hat{V} e_j := z^{1/m} \lambda^j e_j, \quad 1 \leq j \leq m.$$
Show that there is a representation of $A_\lambda(z,w)$ with $U \mapsto \hat{U}$ and $V \mapsto \hat{V}$.
3. Use that m, n are coprime to show that $\{\hat{V}^k \hat{U}^l : 0 \leq k, l < m\}$ span $\mathbb{M}_m(\mathbb{C})$.
4. Deduce that $A_\lambda(z,w) \cong \mathbb{M}_m(\mathbb{C})$ and that $\widehat{A_\lambda} = \mathbb{T}^2$.