Exercise sheet 10.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Monday, 17.6.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise continues the discussion of irrational rotation algebras, now in the irrational case.) Now let $\lambda = e^{2\pi i\theta}$ with irrational θ. Fix $z \in \mathbb{C}$ with $|z| = 1$. Define $\tilde{u}, \tilde{v} \in \mathbb{B}(\ell^2(\mathbb{Z}))$ by

$$\tilde{u}f(n) = f(n-1), \qquad \tilde{v}f(n) = z \cdot \lambda^n \cdot f(n).$$

1. Show that these are unitaries that satisfy the defining relations of the rotation algebra A_λ. So they generate a representation π_z of A_λ.
2. Show that the representation π_z is irreducible.
3. Show that the representations π_z and π_w for another $w \in \mathbb{C}$ with $|w| = 1$ are unitarily equivalent if and only if there is $n \in \mathbb{Z}$ with $z/w = \lambda^n$. Hint: If $z/w = \lambda^n$ for some $n \in \mathbb{Z}$, then a power of $\tilde{u}$ intertwines π_z and π_w. Otherwise, the unitaries $\pi_z(\tilde{v})$ and $\pi_w(\tilde{v})$ have different sets of eigenvalues, so that they are not unitarily equivalent.

For irrational θ, the subgroup $\lambda^{\mathbb{Z}}$ is dense in the unit circle $\mathbb{T}$. So the quotient topology on the space $\mathbb{T}/\lambda^{\mathbb{Z}}$ is the chaotic one. Noncommutative geometry considers A_λ as a C*-algebraic model for this badly behaved quotient space. We have found one justification from this, namely, a rather canonical injective map from $\mathbb{T}/\lambda^{\mathbb{Z}}$ into $\widehat{A_\lambda}$.

Exercise 2. (This exercise describes the space of irreducible representations in an example that is almost commutative in the sense that its centre is quite large.) Let

$$A := \{f \in C([0,1], \mathbb{M}_2(\mathbb{C})) : f(0) \text{ is diagonal}\}.$$

Show that every $[\pi] \in \hat{A}$ vanishes on the ideal $I_{q(\pi)} := \overline{\{f \in Z(A) : q(\pi)(f) = 0\}} \cdot A \subseteq A$ and hence gives a representation of $A_{q(\pi)} := A/I_{q(\pi)}$. Show that $\hat{A} \cong \{[0,1] \times \{a,b\} : (t,a) \sim (t,b),\ \text{for } t \neq 0\}$.

Exercise 3. (This exercise discusses the interaction between Hilbert modules and ideals and quotients.) Let $\mathcal{E}$ be a Hilbert B-module and let $\pi\colon B \twoheadrightarrow A$ be a surjective *-homomorphism. Let $\mathcal{E}_0 := \{x \in \mathcal{E} : \langle x \mid x\rangle_B \in \ker\pi\}$ and $\mathcal{E}_1 := \mathcal{E}/\mathcal{E}_0$. Show that $\mathcal{E}_0$ is a Hilbert module over $\ker\pi$ and that $\mathcal{E}_1$ is a Hilbert A-module in a canonical way.

Exercise 4. (This exercise is the starting point to investigate when the fixed point algebra and the crossed product for an action of a compact group are Morita equivalent.) Let A be a C*-algebra and let G be a finite group with $|G|$ elements. Let α be an action of G on A by automorphisms. Let

$$A^G := \{a \in A : \alpha_g(a) = a \text{ for all } g \in G\}.$$

This is a C*-subalgebra of A.

1. Show that A with the right A^G-module structure by multiplication and with the A^G-valued scalar product
$$\langle a \mid b\rangle := \frac{1}{|G|}\sum_{g\in G} \alpha_g(a^*b)$$
is a pre-Hilbert A-module.
2. Show that the norm defined by this inner product is equivalent to the C*-norm on A and deduce that A is even a Hilbert A^G-module.