Exercise sheet 11.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Monday, 24.6.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This is another important example of multiplier algebras. The result remains true for Hilbert modules over arbitrary C^*-algebras. Feel free to prove the more general statement.) Let $\mathcal{H}$ be a Hilbert space. Prove that $\mathbb{B}(\mathcal{H})$ is naturally isomorphic to the multiplier algebra of $\mathbb{K}(\mathcal{H})$.

Exercise 2. (This exercise is a way to discover the concept of a conditional expectation onto a C^*-subalgebra.) Let $B \subseteq A$ be a C^*-subalgebra and let $E\colon A \to B$ be a map. Which properties must E satisfy, so that A with the obvious right B-module structure and the scalar product $\langle x \mid y \rangle := E(x^*y)$ becomes a pre-Hilbert B-module and the inclusion $B \hookrightarrow A$ becomes an isometry?

Exercise 3. (This exercise describes Hilbert modules in a way that does not require the C^*-algebra of coefficients.) Let $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$ be a closed linear subspace with $x_1 x_2^* x_3 \in \mathcal{E}$ for all $x_1, x_2, x_3 \in \mathcal{E}$ (such spaces are called *ternary rings of operators*). Let $B \subseteq \mathbb{B}(\mathcal{H})$ be the closed linear span of $\{x_2^* x_3 : x_2, x_3 \in \mathcal{E}\}$. Show that B is a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ and that $\mathcal{E}$ is a concrete Hilbert B-module.

Exercise 4. Show that any closed B-submodule $\mathcal{F}$ of a Hilbert module is a Hilbert B-module in its own right, such that the embedding $\mathcal{F} \hookrightarrow \mathcal{E}$ is isometric. Show that

$$\mathcal{F}^\perp := \{\xi \in \mathcal{F} : \xi \in \mathcal{E},\ \langle \xi \mid \eta \rangle = 0 \text{ for all } \eta \in \mathcal{F}\}$$

is a closed B-submodule of $\mathcal{E}$ with $\mathcal{F} \cap \mathcal{F}^\perp = \{0\}$. Show that the embedding $\mathcal{F} \hookrightarrow \mathcal{E}$ is adjointable if and only if $\mathcal{F}$ is complementable.