MATHEMATISCHES INSTITUT
GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN
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NONCOMMUTATIVE GEOMETRY I
SUMMER SEMESTER 2024

Exercise sheet 12.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Monday, 1.7.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise applies the description of hereditary C^*-subalgebras of the compact operators on a Hilbert module.) Let $\mathcal{F} \subseteq \mathcal{E}$ be a closed Hilbert submodule in a Hilbert module. Show that

$$\mathbb{K}(\mathcal{F}) = \{T \in \mathbb{K}(\mathcal{E}) : T(\mathcal{E}) \subseteq \mathcal{F},\ T^*(\mathcal{E}) \subseteq \mathcal{F}\}.$$

Show first that the right hand side is a hereditary C^*-subalgebra of $\mathbb{K}(\mathcal{E})$.

Exercise 2. (This exercise introduces the Cuntz–Toeplitz C^*-algebras, which are related to the Cuntz algebras, but have one relation less.) Fix $n \in \mathbb{N} \cup \{\infty\}$. Let $B = \mathbb{C}$ and let $\mathcal{E} = \mathbb{C}^n$ or $\mathcal{E} = \ell^2(\mathbb{N})$ if $n = \infty$. Call a Toeplitz representation of $\mathcal{E}$ in a unital C^*-algebra D unital if the underlying representation of B is unital. Show that these Toeplitz representations are in bijection with families of isometries $s_1, \dots, s_n$ with orthogonal ranges, that is, subject to the relations $s_j^* s_k = \delta_{j,k}$ for $1 \le j, k \le n$.

Exercise 3. (This is another useful example of a Toeplitz C^*-algebra. The result remains true for nonunital C^*-algebras, but the proof is a bit more difficult.) Let A be a unital C^*-algebra and let α be an automorphism of A. Let A_α be A with the standard Hilbert A-module structure and the left action changed to $a \cdot b := \alpha(a)b$. (This is a C^*-correspondence, and there is no need to prove this unless you really think it helps you.) Show that a unital Toeplitz representation of A in a C^*-algebra D is equivalent to a pair (φ, v) where $\varphi\colon A \to D$ is a unital *-homomorphism and $v \in D$ is an isometry, such that $\varphi(a)v = v\varphi(\alpha(a))$ for all $a \in A$. Show that this relation is equivalent to $\varphi(a)vv^* = v\varphi(\alpha(a))v^*$; this is only compatible with the more familiar relation $\varphi(\alpha^{-1}(a)) = v\varphi(a)v^*$ if v is unitary.

Exercise 4. (This exercise first describes when two morphisms induce isomorphic C^*-correspondences and then examines the vertical product in this special case. With a bit more work, this leads to a 2-category (strict bicategory) that has C^*-algebras as objects, morphisms as arrows, and unitary intertwiners as 2-arrows.) Let A, B, C be C^*-algebras and let $f_1, f_2\colon A \to \mathcal{M}(B)$ and $g_1, g_2\colon B \to \mathcal{M}(C)$ be morphisms. Show that the correspondences associated to f_1, f_2 are isomorphic if and only if there is a unitary multiplier $u \in \mathcal{M}(B)$ with $uf_1(a)u^* = f_2(a)$ for all $a \in A$. Assume this and also a unitary $v \in \mathcal{M}(C)$ with $vg_1(b)v^* = g_2(b)$ for all $b \in B$, giving an isomorphism between the correspondences associated to g_1, g_2. Find two equivalent formulas for a unitary w with $wg_1(f_1(a))w^* = g_2(f_2(a))$.