Mathematisches Institut
Georg-August-Universität Göttingen
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Noncommutative geometry I
Summer semester 2024

Exercise sheet 13.

Name

Exercise group (tutor's name)

Deadline: **Monday, 8.7.2024, 10:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise describes a common source of C^*-correspondences. Actually, the exercise shows that these are even Hilbert bimodules, which is a bit more.) Let B be a C^*-algebra with a continuous circle action β. For $n \in \mathbb{Z}$, let

$$B_n := \{b \in B : \beta_z(b) = z^n b \text{ for all } z \in \mathbb{T}\}.$$

1. Show that $B_0 \subseteq B$ is a C^*-subalgebra.
2. Show that B_n is a B_0-B_0-correspondence with the bimodule structure by multiplication in B and the inner product $\langle \xi \mid \eta \rangle := \xi^* \eta$.
3. Fix $n \in \mathbb{Z}$. Show that the closed linear span of $\xi\eta^*$ for $\xi, \eta \in B_n$ is a closed ideal in B_0, which we denote by I_n.
4. Show that $\xi\eta^*$ for $\xi, \eta \in B_n$ acts on B_n by a rank-one operator. Deduce that the left action of B_0 on B_n restricts to an isomorphism from I_n onto the C^*-algebra of compact operators on B_n.
5. Show that for $n, k \in \mathbb{Z}$ there is an isometric B_0-bimodule map
$$B_n \otimes_{B_0} B_k \to B_{n+k}, \qquad \xi \otimes \eta \mapsto \xi \cdot \eta.$$
6. Now specialise to the case where $B = \mathrm{C}^*(A, \alpha)$ for an automorphism α of a C^*-algebra A, equipped with its canonical gauge action. Show that the maps $B_n \otimes_{B_0} B_k \to B_{n+k}$ are unitary for all $n, k \in \mathbb{Z}$.
7. Now specialise to the case where B is the Toeplitz C^*-algebra of a C^*-correspondence, equipped with its gauge action. Show that the map $B_n \otimes_{B_0} B_k \to B_{n+k}$ is unitary if n, k have the same sign, but not if they have different sign.
8. Now specialise to the case where B is the Cuntz–Pimsner algebra of a C^*-correspondence. Can you find some necessary or sufficient criteria for the map $B_n \otimes_{B_0} B_k \to B_{n+k}$ to be unitary?

Exercise 2. Let V be a discrete set. Let $A = \mathrm{C}_0(V)$ and let $\mathcal{E}$ be an A-A-correspondence. Let $\varphi \colon A \to \mathbb{B}(\mathcal{E})$ be the left action. (This exercise classifies the A-A-correspondences.)

1. For each $v \in V$, let $\chi_v \in \mathrm{C}_0(V)$ be the characteristic function of $\{v\}$ and let $\mathcal{E}_v := \mathcal{E} \cdot \chi_v \subseteq \mathcal{E}$. Show that this becomes a Hilbert space with the inner product $\langle \xi \mid \eta \rangle_v := \langle \xi \mid \eta \rangle_{\mathrm{C}_0(V)}(v)$.
2. Show that $\mathcal{E}$ is the C_0-direct sum of the Hilbert spaces $\mathcal{E}_v$ for $v \in V$:
$$\mathcal{E} = \left\{ (\xi_v)_{v \in V} \in \prod_{v \in V} \mathcal{E}_v : (v \mapsto \|\xi_v\|) \in \mathrm{C}_0(V) \right\}.$$
3. Let $a_{w,v}$ be the dimension of $\chi_w \cdot \mathcal{E} \cdot \chi_v = \chi_w \mathcal{E}_v$. (You may assume that the Hilbert spaces $\mathcal{E}_v$ are all separable if you do not feel familiar with uncountable cardinalities.) Show that two A-A-correspondences are isomorphic if and only if they produce the same dimensions $a_{w,v}$ for all $w, v \in V$.
4. Show that $\varphi(\chi_w) \in \mathbb{B}(\mathcal{E})$ is in $\mathbb{K}(\mathcal{E})$ if and only if the cardinality $\sum_{v \in V} a_{w,v}$ is finite.
5. Show that the ideal $(\operatorname{Ker} \varphi)^\perp \cap \varphi^{-1}(\mathbb{K}(\mathcal{E})) \subseteq \mathrm{C}_0(V)$ is $\mathrm{C}_0(V_{\mathrm{reg}})$, where
$$V_{\mathrm{reg}} := \left\{ v \in V : 0 < \sum_{v \in V} a_{w,v} < \infty \right\}.$$