HILBERT C*-MODULES AND CORRESPONDENCES, CUNTZ–PIMSNER ALGEBRAS

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This is an excerpt of an unfinished book project by me. It is a bit too advanced for this course, but may be useful nevertheless, if you just ignore the things that I did not cover in class.

1. Hilbert C*-modules

We begin by motivating the definition of a Hilbert C*-module. Let A and B be C*-algebras and let $\mathcal{E}$ be an A,B-bimodule. In order to be compatible with the C*-algebra structure on A and B, we want extra structure on $\mathcal{E}$ that allows us to map Hilbert space representations of B to Hilbert space representations of A in a natural way. This should be done by completing $\mathcal{E} \otimes_B \mathcal{H}$ in some "well behaved" inner product if $\mathcal{H}$ is a Hilbert space with a representation $\varphi\colon B \to \mathbb{B}(\mathcal{H})$.

There are two issues: first, we need an inner product on $\mathcal{E} \otimes_B \mathcal{H}$ and, secondly, we need the homomorphism $\psi\colon A \to \operatorname{End}(\mathcal{E} \otimes_B \mathcal{H})$ defined by $\psi(a)(x \otimes y) := (a \cdot x) \otimes y$ to extend to a representation on the Hilbert space completion of $\mathcal{E} \otimes_B \mathcal{H}$. The concepts of a Hilbert B-module and an A,B-correspondence address these two issues. In this section, we focus on the first issue.

Let $\varphi\colon B \to \mathbb{B}(\mathcal{H})$ be a representation. We want to guess a formula for the inner product on $\mathcal{E} \otimes_B \mathcal{H}$. If $x \in \mathcal{E}$, define

$$|x\rangle\colon \mathcal{H} \to \mathcal{E} \otimes_B \mathcal{H}, \qquad y \mapsto x \otimes y. \tag{1.1}$$

the inner product on $\mathcal{E} \otimes_B \mathcal{H}$ should make these operators bounded. Then the adjoint $|x\rangle^*\colon \mathcal{E} \otimes_B \mathcal{H} \to \mathcal{H}$ is defined because $\mathcal{H}$ is a Hilbert space. Thus we get operators $|x_1\rangle^*|x_2\rangle \in \mathbb{B}(\mathcal{H})$ for $x_1, x_2 \in \mathcal{E}$. We write $\langle x| := |x\rangle$ for $x \in \mathcal{E}$. We expect the inner product on $\mathcal{E} \otimes_B \mathcal{H}$ to be "natural". In particular, if U is a unitary in the commutant B' of $B \subseteq \mathbb{B}(\mathcal{H})$, then the induced operator $\mathrm{id} \otimes_B U$ on $\mathcal{E} \otimes_B \mathcal{H}$ should also be unitary. Then

$$\begin{aligned}\langle x_1||x_2\rangle U &= \langle x_1|(\mathrm{id} \otimes_B U)|x_2\rangle = ((\mathrm{id} \otimes_B U^*)|x_1\rangle)^*|x_2\rangle = (|x_1\rangle U^*)^*|x_2\rangle \\ &= U\langle x_1||x_2\rangle.\end{aligned}$$

Then $\langle x_1||x_2\rangle$ belongs to the bicommutant $\varphi(B)''$ for all $x_1, x_2 \in \mathcal{E}$ because the unital C*-algebra $\varphi(B)'$ is spanned by its unitaries. Now we come to our most restrictive assumption: we even assume $\langle x_1||x_2\rangle \in \varphi(B)$ for all $x_1, x_2 \in \mathcal{E}$. If φ is faithful, this gives a map

$$\mathcal{E} \times \mathcal{E} \to B, \qquad (x_1, x_2) \mapsto \langle x_1 \,|\, x_2\rangle := \varphi^{-1}\bigl(|x_1\rangle^*|x_2\rangle\bigr).$$

This B-valued inner product pins down the inner product on $\mathcal{E} \otimes_B \mathcal{H}$ because

$$\bigl\langle x_1 \otimes y_1 \,\big|\, x_2 \otimes y_2\bigr\rangle = \bigl\langle |x_1\rangle y_1 \,\big|\, |x_2\rangle y_2\bigr\rangle = \bigl\langle y_1 \,\big|\, |x_1\rangle^*|x_2\rangle y_2\bigr\rangle = \bigl\langle y_1 \,\big|\, \varphi\bigl(\langle x_1 \,|\, x_2\rangle\bigr) y_2\bigr\rangle$$

for all $x_1, x_2 \in \mathcal{E}$, $y_1, y_2 \in \mathcal{H}$ and the inner product is sesquilinear.

Remark 1.1. The bicommutant $\varphi(B)''$ is the closure of $\varphi(B)$ in the strong operator topology. If $\varphi\colon \mathrm{C}_0(X) \to \mathbb{B}(L^2(X, \mu))$ is a cyclic representation of a commutative C*-algebra, then $\varphi(B)'' = L^\infty(X, \mu)$. So assuming that an element of $\varphi(B)''$ belongs to $\varphi(B)$ is like assuming that a measurable function is continuous.

The B-valued inner product on $\mathcal{E}$ has the following properties. First, it is linear in the second variable and conjugate-linear in the first variable. Secondly, $\langle x_1 \cdot b_1 \,|\, x_2 \cdot b_2\rangle = b_1^* \cdot \langle x_1 \,|\, x_2\rangle \cdot b_2$ for all $x_1, x_2 \in \mathcal{E}$, $b_1, b_2 \in B$ because $|x \cdot b\rangle = |x\rangle\varphi(b)$. Third, $\langle x_1 \,|\, x_2\rangle^* = (|x_1\rangle^*|x_2\rangle)^* = |x_2\rangle^*|x_1\rangle = \langle x_2 \,|\, x_1\rangle$ for all $x_1, x_2 \in \mathcal{E}$. And $\langle x \,|\, x\rangle = |x\rangle^*|x\rangle \geq 0$ for all $x \in \mathcal{E}$. In addition, it is reasonable to assume $|x\rangle \neq 0$ for $x \neq 0$. This is equivalent to $\langle x \,|\, x\rangle \neq 0$ for $x \neq 0$. We turn these properties of our construction into a definition:

Definition 1.2. Let B be a C^*-algebra. A *pre-Hilbert B-module* is a right B-module $\mathcal{E}$ with an inner product $\langle \textvisiblespace \,|\, \textvisiblespace \rangle_B \colon \mathcal{E} \times \mathcal{E} \to B$ such that

(1) $\langle \textvisiblespace \,|\, \textvisiblespace \rangle_B$ is linear in the second variable;
(2) $\langle x_1 \,|\, x_2 \cdot b\rangle_B = \langle x_1 \,|\, x_2\rangle_B \cdot b$ for all $x_1, x_2 \in \mathcal{E}$, $b \in B$;
(3) $\langle x_1 \,|\, x_2\rangle_B^* = \langle x_2 \,|\, x_1\rangle_B$ for all $x_1, x_2 \in \mathcal{E}$;
(4) $\langle x \,|\, x\rangle_B \geq 0$ for all $x \in \mathcal{E}$;
(5) $\langle x \,|\, x\rangle_B \neq 0$ for $x \neq 0$, $x \in \mathcal{E}$.

Conditions (1)–(3) imply that $\langle \textvisiblespace \,|\, \textvisiblespace \rangle_B$ is conjugate-linear in the first variable and satisfies

$$\langle x_1 \cdot b_1 \,|\, x_2 \cdot b_2\rangle_B = b_1^* \cdot \langle x_1 \,|\, x_2\rangle_B \cdot b_2$$

for all $x_1, x_2 \in \mathcal{E}$, $b_1, b_2 \in B$.

We will define a Hilbert B-module as a pre-Hilbert B-module that is complete in a certain natural norm. Before we come to this, we show that the assumptions in Definition 1.2 suffice to define inner products as above:

Proposition 1.3. *Let B be a C^*-algebra, $\mathcal{E}$ a pre-Hilbert B-module, $\mathcal{H}$ a Hilbert space, and $\varphi\colon B \to \mathbb{B}(\mathcal{H})$ a representation. The sesquilinear extension of*

$$\langle x_1 \otimes y_1 \,|\, x_2 \otimes y_2\rangle := \langle y_1 \,|\, \varphi(\langle x_1 \,|\, x_2\rangle_B) y_2\rangle_{\mathcal{H}} \tag{1.2}$$

for $x_1, x_2 \in \mathcal{E}$, $y_1, y_2 \in \mathcal{H}$ is a positive semidefinite inner product on $\mathcal{E} \otimes_B \mathcal{H}$. If φ is faithful, then the conditions (1)–(4) *in Definition* 1.2 *are necessary for this.*

Let $x \in \mathcal{E}$. There is a bounded operator $\mathcal{H} \to \mathcal{E} \otimes_B \mathcal{H}$ with $|x\rangle y := x \otimes y$ for $y \in \mathcal{H}$. It satisfies

$$\big\| |x\rangle \big\| \leq \big\| \langle x \,|\, x\rangle_B \big\|^{1/2},$$

with equality if φ is faithful. And if $x_1, x_2 \in \mathcal{E}$, $y \in \mathcal{H}$, then

$$|x_1\rangle^*(x_2 \otimes y) = \varphi\big(\langle x_1 \,|\, x_2\rangle_B\big) y, \qquad |x_1\rangle^*|x_2\rangle = \varphi\big(\langle x_1 \,|\, x_2\rangle_B\big).$$

Proof. To simplify notation, we assume φ to be faithful. We are going to prove that various properties of the inner product on $\mathcal{E} \otimes_B \mathcal{H}$ are equivalent to similar properties of the B-valued inner product on $\mathcal{E}$. In each case, we only need φ to be faithful to go from the inner product on $\mathcal{E} \otimes_B \mathcal{H}$ to the B-valued inner product on $\mathcal{E}$. So the arguments below prove what is claimed in the proposition also for non-faithful representations φ.

By convention, Hilbert space inner products are linear in the second and conjugate-linear in the first variable. The formula in (1.2) well defines such a sesquilinear map on $\mathcal{E} \otimes \mathcal{H}$ if $\mathcal{E}$ is a pre-Hilbert module. Conversely, (1) in Definition 1.2 is necessary for this to work. The sesquilinear map on $\mathcal{E} \otimes \mathcal{H}$ descends to $\mathcal{E} \otimes_B \mathcal{H}$ if and only if the following holds:

$$\langle x_1 \cdot b \otimes y_1 \,|\, x_2 \otimes y_2\rangle = \langle x_1 \otimes b \cdot y_1 \,|\, x_2 \otimes y_2\rangle,$$
$$\langle x_1 \otimes y_1 \,|\, x_2 \cdot b \otimes y_2\rangle = \langle x_1 \otimes y_1 \,|\, x_2 \otimes b \cdot y_2\rangle.$$

This is equivalent to

$$\langle x_1 \,|\, x_2 \cdot b\rangle_B = \langle x_1 \,|\, x_2\rangle_B \cdot b,$$
$$\langle x_1 \cdot b \,|\, x_2\rangle_B = b^* \cdot \langle x_1 \,|\, x_2\rangle_B.$$

The symmetry property $\langle \xi_1 \,|\, \xi_2 \rangle = \overline{\langle \xi_2 \,|\, \xi_1 \rangle}$ for $\xi_1, \xi_2 \in \mathcal{E} \otimes_B \mathcal{H}$ required for inner products is equivalent to $\langle x_1 \,|\, x_2 \rangle_B^* = \langle x_2 \,|\, x_1 \rangle_B$ for all $x_1, x_2 \in B$. Write $\xi \in \mathcal{E} \otimes_B \mathcal{H}$ as $\xi = \sum_{i=1}^n x_i \otimes y_i$ with $x_i \in \mathcal{E}$, $y_i \in \mathcal{H}$. What does the positivity of $\langle \xi \,|\, \xi \rangle \geq 0$ mean? We expand

$$\langle \xi \,|\, \xi \rangle = \sum_{i,j=1}^{n} \big\langle y_i \,\big|\, \varphi(\langle x_i \,|\, x_j \rangle_B) y_j \big\rangle = \langle Y \,|\, \varphi_n(X) Y \rangle$$

with $Y = (y_1, \dots, y_n) \in \mathcal{H}^n$ and

$$X = \big(\langle x_i \,|\, x_j \rangle_B\big)_{i,j=1,\dots,n} \in \mathbb{M}_n(B).$$

Since φ induces a faithful representation of $\mathbb{M}_n(B)$ on $\mathcal{H}^n$, $\langle \xi \,|\, \xi \rangle \geq 0$ for all $\xi \in \mathcal{E} \otimes_B \mathcal{H}$ if and only if $X \geq 0$ in $\mathbb{M}_n(B)$ for all $x_1, \dots, x_n \in \mathcal{E}$. All these positivity conditions already follow from the case $n = 1$:

Lemma 1.4. *If $\langle x \,|\, x \rangle_B \geq 0$ for all $x \in \mathcal{E}$, then $X \geq 0$ in $\mathbb{M}_n(B)$ for all $x_1, \dots, x_n \in \mathcal{E}$.*

Proof. Let $b_1, \dots, b_n \in B$. Then

$$\begin{aligned} 0 &\leq \langle x_1 b_1 + \dots + x_n b_n \,|\, x_1 b_1 + \dots + x_n b_n \rangle_B \\ &= \sum_{i,j=1}^{n} b_i^* \langle x_i \,|\, x_j \rangle_B b_j = \sum_{i,j=1}^{n} b_i^* X_{ij} b_j. \end{aligned}$$

We claim that a matrix X in $\mathbb{M}_n(B)$ is positive if and only if $\sum_{i,j=1}^n b_i^* X_{ij} b_j \geq 0$ for all $b_1, \dots, b_n \in B$. This is necessary because the sum is $b^* \cdot X \cdot b$ for b the column vector with entries b_i. For sufficiency, we first show that $\psi_n(X) \geq 0$ in $\mathbb{B}(\mathcal{H}^n)$ for any cyclic representation $\psi\colon B \to \mathbb{B}(\mathcal{H})$. Let $y_0 \in \mathcal{H}$ be the cyclic vector. Then $\psi(B) y_0 \subseteq \mathcal{H}$ is dense. If $\eta \in (\psi(B) y_0)^n \subseteq \mathcal{H}^n$, then write $\eta_i = \psi(b_i) y_0$; we compute

$$\langle \eta \,|\, \psi_n(X) \eta \rangle = \sum_{i,j=1}^{n} \langle y_0 \,|\, \psi(b_i^* X_{ij} b_j) y_0 \rangle \geq 0.$$

Then $\psi_n(X) \geq 0$ in $\mathbb{B}(\mathcal{H}^n)$ because vectors of this form are dense in $\mathcal{H}^n$. If B is separable, then it has a faithful state and hence a faithful cyclic representation ψ by the GNS-construction; then the representation ψ_n of $\mathbb{M}_n(B)$ is faithful as well, and we are done. In general, we may decompose a faithful representation of B into cyclic subrepresentations and apply the result to these. Thus $X \geq 0$ in $\mathbb{M}_n(B)$ if $\psi_n(X) \geq 0$ for all cyclic representations ψ. □

So far, we have seen the following: if $\mathcal{E}$ is a pre-Hilbert module, then (1.2) defines a positive semi-definite inner product on $\mathcal{E} \otimes_B \mathcal{H}$, and the converse holds if φ is faithful. Define $|x_1\rangle\colon \mathcal{H} \to \mathcal{E} \otimes_B \mathcal{H}$ as above, and define $\langle x_1|\colon \mathcal{E} \otimes_B \mathcal{H} \to \mathcal{H}$ by linear extension of $\langle x_1|(x_2 \otimes y) := \varphi(\langle x_1 \,|\, x_2 \rangle_B) y$. If $x_1, x_2 \in \mathcal{E}$, $y_1, y_2 \in \mathcal{H}$, then

$$\langle |x_1\rangle y_1 \,|\, x_2 \otimes y_2 \rangle = \langle x_1 \otimes y_1 \,|\, x_2 \otimes y_2 \rangle = \langle y_1 \,|\, \varphi(\langle x_1 \,|\, x_2 \rangle_B) y_2 \rangle = \langle y_1 \,|\, \langle x_1|(x_2 \otimes y_2) \rangle.$$

Hence $\langle x_1| = |x_1\rangle^*$. Thus $|x_1\rangle^* |x_2\rangle = \langle x_1||x_2\rangle = \varphi(\langle x_1 \,|\, x_2 \rangle_B)$. This implies

$$\big\| |x_1\rangle \big\| = \big\| |x_1\rangle^* |x_1\rangle \big\|^{1/2} = \big\| \varphi(\langle x_1 \,|\, x_1 \rangle \big\|^{1/2} \leq \big\| \langle x_1 \,|\, x_1 \rangle_B \big\|^{1/2},$$

with equality if φ is faithful. This finishes the proof of Proposition 1.3. □

Let $\mathcal{E}$ be a pre-Hilbert module. Then $\|\langle x \,|\, x \rangle_B\|^{1/2} = \| |x\rangle \|$ is a norm on $\mathcal{E}$ by Proposition 1.3. This is nontrivial.

Definition 1.5. A *Hilbert B-module* is a pre-Hilbert B-module that is complete in the norm $\|x\| := \|\langle x \,|\, x \rangle_B\|^{1/2}$.

Remark 1.6. Hilbert modules over commutative unital C*-algebras were introduced by Kaplansky [13]. Paschke [22] generalised them to general C*-algebras. They became popular through the work of Rieffel [24].

Exercise 1.7. *If $\mathcal{E}$ is only a pre-Hilbert B-module, then the completion of $\mathcal{E}$ in the norm above is a Hilbert B-module.*

Example 1.8. Hilbert $\mathbb{C}$-modules are the same as Hilbert spaces.

Example 1.9. Any C*-algebra B is a Hilbert module over itself for the standard right module structure and inner product $\langle x\,|\,y\rangle_B := x^*y$ for $x, y \in B$.

Example 1.10. Let B be a C*-algebra and $n \in \mathbb{N}$. Equip B^n with the usual right B-module structure and the inner product

$$\big\langle (x_1,\dots,x_n) \,\big|\, (y_1,\dots,y_n)\big\rangle_B := \sum_{i=1}^n x_i^* y_i. \tag{1.3}$$

This is a Hilbert B-module. It is complete because the norm on B^n from the inner product lies between $\max\{\|b_i\|\}$ and $\sum \|b_i\|$ and B^n is complete in any such norm.

Letting $n \to \infty$, we get a pre-Hilbert module structure on $\bigcup_{n\in\mathbb{N}} B^n$, the algebraic direct sum of countably many copies of B. Its Hilbert module completion is

$$B^\infty := \left\{ (b_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} B : \sum_{i\in\mathbb{N}} b_i^* b_i \text{ converges in } B \right\}. \tag{1.4}$$

This Hilbert module is called the *standard Hilbert B-module* because of Kasparov's Stabilisation Theorem (see Section 10).

Example 1.11. More generally, the direct sum $\mathcal{E}^n$ of n copies of a Hilbert module $\mathcal{E}$ is a Hilbert module with the induced right module structure and the inner product as in (1.3). The argument is the same as for Example 1.10.

Exercise 1.12. *Let $\mathcal{E}$ be a Hilbert B-module and let $\pi\colon B \twoheadrightarrow A$ be a surjective *-homomorphism. Let $\mathcal{E}_0 := \{x \in \mathcal{E} : \langle x\,|\,x\rangle_B \in \ker\pi\}$ and $\mathcal{E}_1 := \mathcal{E}/\mathcal{E}_0$. Show that $\mathcal{E}_0$ is a Hilbert module over $\ker\pi$ and that $\mathcal{E}_1$ is a Hilbert A-module in a canonical way.*

Exercise 1.13. *Let X be a locally compact space. A* continuous field of Hilbert spaces *is a family of Hilbert spaces $(\mathcal{H}_x)_{x\in X}$ with a subspace $\mathcal{H} \subseteq \prod_{x\in X} \mathcal{H}_x$, whose elements are called continuous sections, such that*

- *$\mathcal{H}$ is closed under pointwise multiplication by $\mathrm{C}_0(X)$;*
- *if $\xi, \eta \in \mathcal{H}$, then the function $x \mapsto \langle \xi_x\,|\,\eta_x\rangle$ belongs to $\mathrm{C}_0(X)$;*
- *$\mathcal{H} \subseteq \prod_{x\in X} \mathcal{H}_x$ is closed in the norm $\|(\xi_x)_{x\in X}\| := \sup \|\xi_x\|$;*
- *for any $x \in X$, the evaluation map $\mathcal{H} \to \mathcal{H}_x$ is surjective.*

(*Compare the concept of a continuous field of* C*-*algebras.*) *Show that $\mathcal{H}$ with the pointwise multiplication by $\mathrm{C}_0(X)$ and the pointwise inner product is a Hilbert $\mathrm{C}_0(X)$-module. Show that any Hilbert $\mathrm{C}_0(X)$-module is isomorphic to one of this form for a continuous field of Hilbert spaces, which is unique up to isomorphism.* (*Use Exercise* 1.12 *to construct the fibres $\mathcal{H}_x$.*)

Remark 1.14. If $\mathcal{E}$ is a Hilbert B-module, then $\langle \xi\,|\,\xi\rangle \neq 0$ for $\xi \neq 0$ in $\mathcal{E} \otimes_B \mathcal{H}$ (see Proposition 5.13). It is unclear whether this still holds if $\mathcal{E}$ is a pre-Hilbert B-module. Thus the nondegeneracy of the inner product on $\mathcal{E}$ is necessary and almost sufficient for the inner product on $\mathcal{E} \otimes_B \mathcal{H}$ to be positive definite.

Remark 1.15. Let $\mathcal{E}$ be a B-module with a B-valued inner product that satisfies the conditions (1)–(4) in Definition 1.2, but not (5). Then the proof of Proposition 1.3 goes through without change. But now $\|x\| := \|\langle x\,|\,x\rangle\|^{1/2}$ is only a seminorm on $\mathcal{E}$. Its null space is the subset of all $x \in \mathcal{E}$ with $\langle x\,|\,x\rangle_B = 0$. It follows that this is a right B-submodule in $\mathcal{E}$. In addition, the B-valued inner product on $\mathcal{E}$ descends to the quotient $\mathcal{E}/\mathcal{E}_0$, and this is an honest pre-Hilbert module. Since the map $\mathcal{E} \otimes_B \mathcal{H} \to \mathcal{E}/\mathcal{E}_0 \otimes_B \mathcal{H}$ induced by the quotient map preserves the inner products, $\mathcal{E} \otimes_B \mathcal{H}$ and $\mathcal{E}/\mathcal{E}_0 \otimes_B \mathcal{H}$ have the same Hilbert space completion. Therefore, we gain only irrelevant generality by allowing degenerate B-valued inner products.

Proposition 1.3 shows that any Hilbert module may be realised concretely through a space of operators. This is a powerful tool to reduce statements about Hilbert modules to statements about Hilbert space operators and their operator norms. It is worthwhile to turn this into a definition:

Definition 1.16. Let $\varphi\colon B \to \mathbb{B}(\mathcal{H})$ be a faithful representation. A *concrete Hilbert B-module* relative to φ is a norm-closed subspace $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$ for a Hilbert space $\mathcal{K}$, such that $\mathcal{E}\varphi(B) \subseteq \mathcal{E}$ and $\mathcal{E}^*\mathcal{E} \subseteq \varphi(B)$. A concrete Hilbert B-module is *nondegenerate* if $\mathcal{E}\mathcal{H}$, the closed linear span of $x(y)$ for $x \in \mathcal{E}$, $y \in \mathcal{H}$, is equal to $\mathcal{K}$.

If $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$ is a concrete Hilbert B-module, then the obvious right B-module structure and the inner product $\langle x_1 \,|\, x_2 \rangle_B := x_1^* \circ x_2$ make it a Hilbert B-module.

Lemma 1.17. *Any Hilbert module $\mathcal{E}$ is isomorphic to a concrete one.*

Proof. Let $\varphi\colon B \to \mathbb{B}(\mathcal{H})$ be any faithful representation. Let $\mathcal{K}$ be the completion of $\mathcal{E} \otimes_B \mathcal{H}$ in the inner product (1.2). Proposition 1.3 gives an isometric linear map $\mathcal{E} \hookrightarrow \mathbb{B}(\mathcal{H}, \mathcal{K})$, $x \mapsto |x\rangle$. Since $\mathcal{E}$ is complete, its image is norm-closed. And $|x \cdot b\rangle = |x\rangle\varphi(b)$ and $|x_1\rangle^*|x_2\rangle = \varphi(\langle x_1 \,|\, x_2\rangle_B) \in \varphi(B)$ for all $x, x_1, x_2 \in \mathcal{E}$, $b \in B$. Thus $\{|x\rangle : x \in \mathcal{E}\}$ is a concrete Hilbert B-module, and it defines the Hilbert B-module structure on $\mathcal{E}$. □

Exercise 1.18. *Let $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$ be a closed linear subspace with $x_1 x_2^* x_3 \in \mathcal{E}$ for all $x_1, x_2, x_3 \in \mathcal{E}$ (such spaces are called* ternary rings of operators *and were first studied in* [25]*). Let $B \subseteq \mathbb{B}(\mathcal{H})$ be the closed linear span of $\{x_2^* x_3 : x_2, x_3 \in \mathcal{E}\}$. Show that B is a* C*-*subalgebra of $\mathbb{B}(\mathcal{H})$ and that $\mathcal{E}$ is a concrete Hilbert B-module.*

The triangle inequality for the norm $\|\langle x \,|\, x\rangle_B\|^{1/2}$ on a Hilbert module is usually proven using a generalisation of the Cauchy–Schwarz inequality to Hilbert modules. We have followed a different route. To relate the two proofs, we formulate the Cauchy–Schwarz inequality in Proposition 1.19 and assert in Exercise 1.20 that it is weaker than Lemma 1.4, which is a key step in our proof.

Proposition 1.19. *Let $\mathcal{E}$ be a pre-Hilbert module and $x_1, x_2 \in \mathcal{E}$. Then*

$$\langle x_1 \,|\, x_2\rangle_B \cdot \langle x_1 \,|\, x_2\rangle_B^* \le \|\langle x_2 \,|\, x_2\rangle_B\| \cdot \langle x_1 \,|\, x_1\rangle_B$$

and $\|\langle x_1 \,|\, x_2\rangle_B\| \le \|x_1\|\|x_2\|$. Both inequalities are called Cauchy–Schwarz inequality*.*

Proof. We may realise $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$ as a concrete Hilbert module by choosing a faithful representation of B on a Hilbert space $\mathcal{H}$ and taking $\mathcal{K} := \mathcal{E} \otimes_B \mathcal{H}$. Then $\langle x_1 \,|\, x_2\rangle_B = x_1^* x_2$ in $\mathbb{B}(\mathcal{H}, \mathcal{H})$ and so

$$\langle x_1 \,|\, x_2\rangle_B \cdot \langle x_1 \,|\, x_2\rangle_B^* = x_1^* x_2 x_2^* x_1 \le x_1^* \|x_2 x_2^*\| x_1 = \|\langle x_2 \,|\, x_2\rangle_B\| \cdot \langle x_1 \,|\, x_1\rangle_B.$$

Then $\|\langle x_1 \,|\, x_2\rangle_B\| \le \|x_1\|\|x_2\|$ follows using the C*-identity for B. □

Exercise 1.20 (see [3])**.** *The case $n = 2$ of Lemma 1.4 says that*

$$\begin{pmatrix} \langle x_1 \,|\, x_1\rangle_B & \langle x_1 \,|\, x_2\rangle_B \\ \langle x_2 \,|\, x_1\rangle_B & \langle x_2 \,|\, x_2\rangle_B \end{pmatrix} \ge 0$$

in $\mathbb{M}_2(B)$ for all $x_1, x_2 \in \mathcal{H}$. Show that this implies the Cauchy–Schwarz inequality.

Remark 1.21. I am not aware of a source that uses concrete Hilbert modules systematically to prove basic results about Hilbert modules. A more general concept of a concrete Hilbert module is introduced in [18]. The goal in [18] is to concretely represent Hilbert modules over the reduced crossed product C*-algebra $A \rtimes_{\mathrm{r}} G$ for a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$ of a locally compact group G. Operators between Hilbert A-modules are used for this purpose. So the purpose in [18] is to reduce Hilbert modules over a complicated C*-algebra such as $A \rtimes_{\mathrm{r},\alpha} G$ to Hilbert modules over a simpler C*-algebra such as the coefficient algebra A, but not to develop the theory of Hilbert modules from scratch.

2. Correspondences between C*-algebras

Let $\mathcal{E}$ be an A,B-bimodule with a B-valued inner product that makes it a Hilbert B-module. Let $\mathcal{H}$ be a Hilbert space with a representation $\varphi\colon B \to \mathbb{B}(\mathcal{H})$. We now change our notation slightly: *we write $\mathcal{E} \odot_B \mathcal{H}$ for the algebraic tensor product.* In Section 1, we have equipped $\mathcal{E} \odot_B \mathcal{H}$ with a natural inner product. *From now on, we let $\mathcal{E} \otimes_B \mathcal{H}$ be the Hilbert space completion of $\mathcal{E} \odot_B \mathcal{H}$.*

The bimodule structure on $\mathcal{E}$ gives an algebra homomorphism

$$\psi\colon A \to \mathrm{End}(\mathcal{E} \odot_B \mathcal{H}), \qquad \psi(a)(x \otimes y) := (a \cdot x) \otimes y.$$

We want this to extend to a representation of A on the completion $\mathcal{E} \otimes_B \mathcal{H}$. In particular, we want $\psi(a^*)$ and $\psi(a)$ for $a \in A$ to be adjoints of one another. This happens if and only if

$$\bigl\langle (a \cdot x_1) \otimes y_1 \,\big|\, x_2 \otimes y_2 \bigr\rangle = \bigl\langle x_1 \otimes y_1 \,\big|\, (a^* \cdot x_2) \otimes y_2 \bigr\rangle$$

for all $x_1, x_2 \in \mathcal{E}$, $y_1, y_2 \in \mathcal{H}$. And this is equivalent to

$$\langle a x_1 \,|\, x_2 \rangle_B = \langle x_1 \,|\, a^* x_2 \rangle_B \tag{2.1}$$

for all $x_1, x_2 \in \mathcal{E}$, $a \in A$, if φ is faithful.

Lemma 2.1. *If* (2.1) *holds, then the homomorphism $\psi\colon A \to \mathrm{End}(\mathcal{E} \odot_B \mathcal{H})$ extends to a *-homomorphism $A \to \mathbb{B}(\mathcal{E} \otimes_B \mathcal{H})$. It is nondegenerate if $A \cdot \mathcal{E}$ is norm-dense in $\mathcal{E}$.*

Proof. We replace A by A^+ to make it unital. Any A-module structure extends to A^+ by letting $1 \in A^+$ act by the identity map, and (2.1) still holds for $a \in A^+$. If $u \in A^+$ is unitary, then $\langle u x_1 \,|\, u x_2 \rangle_B = \langle x_1 \,|\, u^* u x_2 \rangle_B = \langle x_1 \,|\, x_2 \rangle_B$. Hence the invertible operator $u \odot_B \mathrm{id}_{\mathcal{H}}$ on $\mathcal{E} \odot_B \mathcal{H}$ is unitary for our inner product. So it induces a unitary operator on $\mathcal{E} \otimes_B \mathcal{H}$. Any element of a unital C*-algebra such as A^+ is a linear combination of unitaries. Therefore, for any $a \in A^+$ there is a bounded linear operator on $\mathcal{E} \otimes_B \mathcal{H}$ that acts by $x \otimes y \mapsto (ax) \otimes y$ on the image of $\mathcal{E} \odot_B \mathcal{H}$. On the pre-Hilbert space $\mathcal{E} \odot_B \mathcal{H}$, the left action of A is multiplicative, and (2.1) gives compatibility with inner products. This remains true on the Hilbert space $\mathcal{E} \otimes_B \mathcal{H}$. So the map $A \to \mathbb{B}(\mathcal{E} \otimes_B \mathcal{H})$ is a *-homomorphism. Since $\mathcal{E} \odot_B \mathcal{H}$ is dense in $\mathcal{E} \otimes_B \mathcal{H}$, $A \cdot \mathcal{E} \otimes_B \mathcal{H}$ is dense in $\mathcal{E} \otimes_B \mathcal{H}$ if $A \cdot \mathcal{E}$ is dense in $\mathcal{E}$. □

Remark 2.2. If $A \cdot \mathcal{E}$ is not dense in $\mathcal{E}$, then there are a Hilbert space $\mathcal{H}$ and a representation $\varphi\colon B \to \mathbb{B}(\mathcal{H})$ such that $(A \cdot \mathcal{E}) \otimes_B \mathcal{H}$ is not dense in $\mathcal{E} \otimes_B \mathcal{H}$ (see Theorem 7.14 below). Then the representation of A on $\mathcal{E} \otimes_B \mathcal{H}$ is degenerate.

Definition 2.3. An *A,B-correspondence* is a Hilbert B-module $\mathcal{E}$ with a nondegenerate left A-module structure satisfying (2.1). By convention, we think of a correspondence as an arrow from B to A and write $\mathcal{E}\colon A \xleftarrow{\mathfrak{C}} B$ if $\mathcal{E}$ is an A,B-correspondence.

Remark 2.4. The assumption (2.1) implies $a(xb) = (ax)b$ for all $a \in A$, $x \in \mathcal{E}$, $b \in B$ (see Lemma 3.2). So A,B-correspondences are A,B-bimodules.

If $S\colon \mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ is an isometric intertwiner between two representations of B, then $\mathrm{id}_{\mathcal{E}} \odot_B S\colon \mathcal{E} \odot_B \mathcal{H}_1 \hookrightarrow \mathcal{E} \odot_B \mathcal{H}_2$ is an isometric A-module map. Therefore, it extends to an isometric intertwiner

$$\mathrm{id}_{\mathcal{E}} \otimes_B S\colon \mathcal{E} \otimes_B \mathcal{H}_1 \hookrightarrow \mathcal{E} \otimes_B \mathcal{H}_2$$

for the induced representations of A. The trick in the proof of Lemma 2.1 shows that the same remains true for bounded intertwiners. We have shown:

Proposition 2.5. *Let $\mathcal{E}$ be an A, B-correspondence. There are functors $\mathcal{E} \otimes_B \sqcup$ between the categories of Hilbert space representations of A and B with intertwining unitaries, isometries, or bounded linear operators as arrows.*

To construct this functor, we needed all the conditions in Definition 1.2 and Definition 2.3 except condition (5) in Definition 1.2 and the completeness of $\mathcal{E}$. These two unnecessary conditions do not reduce generality because we may arrange them by Hausdorff completing $\mathcal{E}$.

Example 2.6. Let $\mathbb{T}'$ be $\mathbb{T}$ as a set, but topologised as the disjoint union of $\mathbb{T} \setminus \{1\}$ and the point $\{1\}$. The identity map is continuous as a map $\mathbb{T}' \to \mathbb{T}$, but not the other way around. So $\mathrm{C}(\mathbb{T}) \subseteq \mathrm{C}_{\mathrm{b}}(\mathbb{T}') = \mathcal{M}(\mathrm{C}_0(\mathbb{T}'))$, and any Hilbert space representation of $\mathrm{C}_0(\mathbb{T}')$ determines one of $\mathrm{C}(\mathbb{T})$. Conversely, a Hilbert space representation of $\mathrm{C}(\mathbb{T})$ extends uniquely to Borel functions on $\mathbb{T}$. These are the same as the Borel functions on $\mathbb{T}'$. So we get a *-homomorphism on $\mathrm{C}_0(\mathbb{T}')$. The Borel functional calculus is built so that $\mathrm{C}_0(\mathbb{T} \setminus \{1\}) \subseteq \mathrm{C}_0(\mathbb{T}')$ acts in the same way as it does as an ideal in $\mathrm{C}(\mathbb{T})$ and the characteristic function of $\{1\}$ acts by the projection to the corresponding eigenspace of the representation. This representation of $\mathrm{C}_0(\mathbb{T}')$ is *nondegenerate*. The usual universal representation $i\colon \mathbb{Z} \to \mathrm{C}^*(\mathbb{Z}) \cong \mathrm{C}(\mathbb{T})$ also defines a representation $i'\colon \mathbb{Z} \to \mathcal{U}(\mathrm{C}_0(\mathbb{T}'))$ because of the inclusion morphism $\mathrm{C}(\mathbb{T}) \rightarrowtail \mathrm{C}_0(\mathbb{T}')$. Let U be a representation of $\mathbb{Z}$ on a Hilbert space $\mathcal{H}$. The functional calculus for $U(1)$ is the unique representation π of $\mathrm{C}(\mathbb{T})$ on $\mathcal{H}$ with $U = \pi_*(i)$. The corresponding representation π' of $\mathrm{C}_0(\mathbb{T}')$ on $\mathcal{H}$ satisfies $U = \pi'_*(i')$. And it is the only representation with that property. So the representation i' of $\mathbb{Z}$ in $\mathrm{C}_0(\mathbb{T}')$ has the property that any Hilbert space representation of $\mathbb{Z}$ is of the form $\pi_*(i')$ for a unique representation π of $\mathrm{C}_0(\mathbb{T}')$. Nevertheless, $\mathrm{C}_0(\mathbb{T}')$ is not isomorphic to the group C*-algebra of $\mathbb{Z}$.

Remark 2.7. The equivalence from the Hilbert space representation category of $\mathrm{C}(\mathbb{T})$ to that of $\mathrm{C}(\mathbb{T}')$ in Example 2.6 does not come from any $\mathrm{C}(\mathbb{T}'), \mathrm{C}(\mathbb{T})$-correspondence. The analogy to the purely algebraic theory of Morita equivalence for rings breaks down at this point. In the C*-algebraic setting, the interesting concept is what Rieffel calls "strong Morita equivalence". Beer [4] gives many more examples of equivalences of Hilbert space representation categories that do not come from a C*-correspondence.

Definition 2.8. Let A and B be C*-algebras, let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $\alpha\colon A \to \mathbb{B}(\mathcal{K})$ and $\beta\colon B \hookrightarrow \mathbb{B}(\mathcal{H})$ be representations, where β is faithful. A *concrete correspondence* $A \overset{\mathcal{E}}{\leftarrow} B$ is a norm-closed subspace $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$ such that $\alpha(a)x, x\beta(b) \in \mathcal{E}$ and $x^*y \in \beta(B)$ for all $a \in A$, $x, y \in \mathcal{E}$, $b \in B$, and $A\mathcal{E} = \mathcal{E}$.

A concrete correspondence is a correspondence with the obvious bimodule structure and inner product. Up to isomorphism, any correspondence is of this form:

Lemma 2.9. *Any correspondence $\mathcal{E}\colon A \overset{\mathcal{E}}{\leftarrow} B$ is isomorphic to a concrete correspondence, where any faithful representation $\beta\colon B \to \mathbb{B}(\mathcal{H})$ may be chosen. In addition, we may arrange that α is faithful or the concrete correspondence is nondegenerate (as a concrete Hilbert B-module). It may be impossible to get both extra properties together.*

Proof. Realise $\mathcal{E}$ as a concrete correspondence in $\mathbb{B}(\mathcal{H}, \mathcal{E} \otimes_B \mathcal{H})$ as in the proof of Lemma 1.17. Lemma 2.1 gives a representation $\alpha\colon A \to \mathbb{B}(\mathcal{E} \otimes_B \mathcal{H})$ with $\alpha(a)(x \otimes y) = (ax) \otimes y$ for all $a \in A$, $x \in \mathcal{E}$, $y \in \mathcal{H}$. This identifies $\mathcal{E}$ with a nondegenerate concrete A,B-correspondence in $\mathbb{B}(\mathcal{H}, \mathcal{E} \otimes_B \mathcal{H})$. We may arrange for α to be faithful by taking $(\mathcal{E} \otimes_B \mathcal{H}) \oplus \mathcal{K}$ with a faithful representation of A on $\mathcal{K}$ and by realising $\mathcal{E}$ through the operators $|x\rangle \oplus 0\colon \mathcal{H} \to (\mathcal{E} \otimes_B \mathcal{H}) \oplus \mathcal{K}$. If $\mathcal{E} = 0$, then $\mathcal{E}\mathcal{H} = 0$ and so nondegeneracy is incompatible with α being faithful. $\square$

Example 2.10. Hilbert $\mathbb{C}$-modules are the same as Hilbert spaces. A correspondence $A \leftarrow \mathbb{C}$ is a Hilbert space representation of A, with the usual nondegeneracy condition. A correspondence $\mathbb{C} \leftarrow B$ is just a Hilbert B-module: the only nondegenerate action of $\mathbb{C}$ is through scalar multiples of the identity map.

Example 2.11. Let $f\colon A \to B$ be a morphism. Then B with the right Hilbert B-module structure from Example 1.9 and the left A-module structure defined by $a \cdot b := f(a)b$ is an A,B-correspondence because

$$\langle a \cdot b_1 \,|\, b_2\rangle := (f(a)b_1)^* b_2 = b_1^*(f(a)^* b_2) = \langle b_1 \,|\, a^* \cdot b_2\rangle$$

for all $a \in A$, $b_1, b_2 \in B$.

3. Adjointable and compact operators on Hilbert modules

A representation of a C*-algebra A on a Hilbert space $\mathcal{H}$ is the same as a morphism $A \to \mathbb{K}(\mathcal{H})$. We are going to introduce a C*-algebra of compact operators on a Hilbert module so that the same is true for representations on Hilbert modules, that is, for correspondences. First we describe the analogue of the C*-algebra of bounded operators on a Hilbert space. For Hilbert module operators, the existence of an adjoint is not automatic (see Example 3.4 below). Since we want a C*-algebra of operators, we only allow operators that are adjointable in the following sense:

Definition 3.1. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert B-modules. A map $f\colon \mathcal{E}_1 \to \mathcal{E}_2$ is *adjointable* if there is a map $f^*\colon \mathcal{E}_2 \to \mathcal{E}_1$, called its *adjoint*, with $\langle x \,|\, f(y)\rangle = \langle f^*(x) \,|\, y\rangle$ for all $x \in \mathcal{E}_2$, $y \in \mathcal{E}_1$. Let $\mathbb{B}(\mathcal{E}_1, \mathcal{E}_2)$ be the set of adjointable operators from $\mathcal{E}_1$ to $\mathcal{E}_2$.

Lemma 3.2. *If f is an adjointable operator then f and f^* are bounded, linear B-module homomorphisms.*

Proof. Let $x_1, x_2, y \in \mathcal{E}$, $b \in B$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}\langle y \,|\, f(x_1 + \lambda x_2 b)\rangle &= \langle f^* y \,|\, x_1 + \lambda x_2 b\rangle \\ &= \langle f^* y \,|\, x_1\rangle + \lambda\langle f^* y \,|\, x_2\rangle b = \langle y \,|\, f(x_1) + \lambda f(x_2) b\rangle.\end{aligned}$$

That is, f is linear and B-linear. We use the Closed Graph Theorem to show that f is bounded: it suffices to prove that its graph is closed. Let $x_n \to x$ and $f(x_n) \to y$ in $\mathcal{E}$ and let $z \in \mathcal{E}$. Then

$$\begin{aligned}\|\langle y - f(x) \,|\, z\rangle\| &= \lim \|\langle f(x_n - x) \,|\, z\rangle\| \\ &= \lim \|\langle x_n - x \,|\, f^* z\rangle\| \le \lim \|x_n - x\| \cdot \|f^* z\| = 0.\end{aligned}$$

This implies $\|f(x) - y\| = 0$ and then $f(x) = y$. $\square$

Proposition 3.3. *The adjointable operators form a unital* C*-*algebra* $\mathbb{B}(\mathcal{E})$ *with the operator norm* $\|f\| := \sup\{\|f(x)\| : x \in \mathcal{E},\ \|x\| = 1\}$.

Proof. If f_1, f_2 are adjointable operators and $\lambda_1, \lambda_2 \in \mathbb{C}$, then $\lambda_1 f_1 + \lambda_2 f_2$ is adjointable with adjoint $\overline{\lambda_1} f_1^* + \overline{\lambda_2} f_2^*$, and $f_1 f_2$ is adjointable with adjoint $f_2^* f_1^*$. If f is adjointable, then so is f^* with $f^{**} = f$. The identity operator is its own adjoint. Thus $\mathbb{B}(\mathcal{E})$ is a unital *-algebra. The operator norm on $\mathbb{B}(\mathcal{E})$ is submultiplicative.

If $x, y \in \mathcal{E}$, then $\|\langle y \mid x\rangle\| \le \|x\| \cdot \|y\|$ because of the Cauchy–Schwarz inequality in Proposition 1.19. Taking $y = \|x\|^{-1/2} \cdot x$ shows that

$$\|x\| = \sup\{\|\langle y \mid x\rangle\| : y \in \mathcal{E},\ \|y\| \le 1\}.$$

This implies

$$\|f\| := \sup\{\|f(x)\| : x \in \mathcal{E},\ \|x\| \le 1\} = \sup\{\|\langle y \mid f(x)\rangle\| : x, y \in \mathcal{E},\ \|x\|, \|y\| \le 1\}.$$

This symmetric expression for the operator norm of f implies $\|f\| = \|f^*\|$. Since the operator norm is submultiplicative, this implies $\|f^*f\| \le \|f^*\| \cdot \|f\| = \|f\|^2$. The following computation gives the other half of the C*-identity:

$$\|f^*f\| = \sup\{\|\langle y \mid f^*fx\rangle\|\} = \sup\{\|\langle fy \mid fx\rangle\|\} \ge \sup\{\|\langle fx \mid fx\rangle\|\} = \|f\|^2.$$

Here the suprema run over $x, y \in \mathcal{E}$ with $\|x\|, \|y\| \le 1$ as above.

Finally, we prove that $\mathbb{B}(\mathcal{E})$ is complete. If (f_n) is a Cauchy sequence, then so is (f_n^*) because $\|f\| = \|f^*\|$. Hence both (f_n) and (f_n^*) converge to linear operators f and f^*, respectively. By the Cauchy–Schwarz inequality, $\langle x \mid f_n y\rangle$ and $\langle f_n^* x \mid y\rangle$ are also Cauchy sequences, which converge to $\langle x \mid fy\rangle$ and $\langle f^*x \mid y\rangle$, respectively. Then

$$\langle x \mid fy\rangle = \lim \langle x \mid f_n y\rangle = \lim \langle f_n^* x \mid y\rangle = \langle f^* x \mid y\rangle.$$

That is, f^* is adjoint to f. Hence $\lim f_n = f$ in $\mathbb{B}(\mathcal{E})$, proving completeness. □

There are bounded B-module homomorphisms that are not adjointable:

Example 3.4. Let $B = \mathrm{C}[0,1]$, $\mathcal{E}_1 = \mathrm{C}_0(0,1]$, $\mathcal{E}_2 = B$ with the obvious Hilbert B-module structures. The inclusion map $\iota\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ is a bounded linear B-module homomorphism, even isometric; but it has no adjoint. To see this, assume that $\iota^*\colon \mathcal{E}_2 \to \mathcal{E}_1$ were an adjoint; let $\xi \in \mathcal{E}_1$ be the function $x \mapsto x$; then

$$x = \mathrm{ev}_x(\langle \xi \mid 1\rangle_B) = \mathrm{ev}_x(\langle \iota(\xi) \mid 1\rangle) = \mathrm{ev}_x(\langle \xi \mid \iota^*(1)\rangle) = x \cdot (\iota^*1)(x);$$

thus $(\iota^*1)(x) = 1$ for all $x \neq 0$, which contradicts $\iota^*1 \in \mathrm{C}_0(0,1]$.

To get an example with $\mathcal{E}_1 = \mathcal{E}_2$, take $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2$ and the operator

$$T\colon \mathcal{E} \to \mathcal{E}, \qquad T(x_1, x_2) := (0, x_1).$$

Definition 3.5. Let $\mathcal{F}$ be a closed B-submodule of a Hilbert B-module $\mathcal{E}$. The *orthogonal complement* of $\mathcal{F}$ is the set

$$\mathcal{F}^\perp := \{y \in \mathcal{E} : \langle x \mid y\rangle = 0 \text{ for all } x \in \mathcal{F}\}.$$

We call $\mathcal{F}$ *complementable* if $\mathcal{F} + \mathcal{F}^\perp = \mathcal{E}$.

Exercise 3.6. *Show that any closed B-submodule $\mathcal{F}$ of a Hilbert module is a Hilbert B-module in its own right, such that the embedding $\mathcal{F} \hookrightarrow \mathcal{E}$ is isometric. Show that $\mathcal{F}^\perp$ is a closed B-submodule of $\mathcal{E}$ with $\mathcal{F} \cap \mathcal{F}^\perp = \{0\}$. Show that the embedding $\mathcal{F} \hookrightarrow \mathcal{E}$ is adjointable if and only if $\mathcal{F}$ is complementable.*

Definition 3.7. Let $\mathcal{E}$ be a pre-Hilbert B-module. For $x \in \mathcal{E}$, define

$$\begin{aligned} \langle x| \colon \mathcal{E} \to B, &\qquad z \mapsto \langle x \mid z\rangle_B, \\ |x\rangle \colon B \to \mathcal{E}, &\qquad b \mapsto x \cdot b. \end{aligned}$$

Remark 3.8. This notation clashes with the definition of $|x\rangle$ in (1.1), which we shall not use any more from now on. This confusion is not so bad because the two uses of the notation $|x\rangle$ are very closely related. Let $\varphi\colon B \to \mathbb{B}(\mathcal{H})$ be a faithful representation. Then $B \otimes_B \mathcal{H} \cong \mathcal{H}$ when we view B as a Hilbert B-module as in Example 1.9. The operators $\langle x| \colon \mathcal{E} \to B$ and $|x\rangle \colon B \to \mathcal{E}$ in Definition 3.7 induce operators

$$|x\rangle \otimes_B 1 \colon \mathcal{H} \cong B \otimes_B \mathcal{H} \leftrightarrow \mathcal{E} \otimes_B \mathcal{H} \colon \langle x| \otimes_B 1$$

(compare Theorem 3.16). These are the operators $|x\rangle^*$ and $|x\rangle$ as in (1.1).

Lemma 3.9. *With the notation $l_a(b) := a \cdot b$, the following formulas hold:*

$$\begin{aligned} |\lambda_1 x_1 + \lambda_2 x_2\rangle &= \lambda_1 |x_1\rangle + \lambda_2 |x_2\rangle, & \langle \lambda_1 x_1 + \lambda_2 x_2| &= \overline{\lambda_1}\langle x_1| + \overline{\lambda_2}\langle x_2|,\\ |x \cdot a\rangle &= |x\rangle l_a, & \langle x \cdot a| &= l_{a^*}\langle x|,\\ |T(x)\rangle &= T|x\rangle, & \langle T(x)| &= \langle x|T^*,\\ |x\rangle^* &= \langle x|, & \langle x_1||x_2\rangle &= l_{\langle x_1 \,|\, x_2\rangle_B}. \end{aligned}$$

Proof. All claims are direct computations. As a sample, we check $|x\rangle^* = \langle x|$:

$$\big\langle y \,\big|\, |x\rangle b\big\rangle_B = \big\langle y \,\big|\, xb\big\rangle_B = \big\langle y \,\big|\, x\big\rangle_B b = \big\langle \langle x \,|\, y\rangle_B \,\big|\, b\big\rangle = \big\langle \langle x|y \,\big|\, b\big\rangle_B. \qquad \square$$

Definition 3.10. For Hilbert B-modules $\mathcal{E}$ and $\mathcal{F}$, let $\mathbb{K}(\mathcal{E},\mathcal{F}) \subseteq \mathbb{B}(\mathcal{E},\mathcal{F})$ be the closed linear span of the operators $|x\rangle\langle y|$ with $x \in \mathcal{F}$, $y \in \mathcal{E}$. Let $\mathbb{K}(\mathcal{E}) := \mathbb{K}(\mathcal{E},\mathcal{E})$.

By Lemma 3.9, $\mathbb{K}(\mathcal{E})$ is a two-sided *-ideal in $\mathbb{B}(\mathcal{E})$.

Example 3.11. If $B = \mathbb{C}$, then $\mathcal{E}$ and $\mathcal{F}$ are Hilbert spaces; the operators of the form $|x\rangle\langle y|$ are the rank-one operators. Taking the linear span yields the finite-rank operators. Hence $\mathbb{K}(\mathcal{E},\mathcal{F})$ is the closure of the space of finite-rank operators.

Proposition 3.12 ([5, Lemme 1.3])**.** *For any $x \in \mathcal{E}$ there is $y \in \mathcal{E}$ with $x = y\langle y \,|\, y\rangle_B = |y\rangle\langle y|y$.*

Proof. Realise $\mathcal{E}$ as a concrete Hilbert module $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H},\mathcal{K})$ by Lemma 2.9. Thus $y\langle y \,|\, y\rangle_B = yy^*y$ for all $y \in \mathcal{E} \subseteq \mathbb{B}(\mathcal{H},\mathcal{K})$. For $\varepsilon > 0$, $x \cdot (x^*x + \varepsilon)^{-1/3}$ exists and belongs to $\mathcal{E} \cdot B^+ = \mathcal{E}$ because $x^*x + \varepsilon$ is an invertible element in the unital C*-algebra B^+. The net $x \cdot (x^*x + \varepsilon)^{-1/3}$ for $\varepsilon \searrow 0$ is norm-Cauchy; hence it converges to some $y \in \mathcal{E}$. This satisfies

$$yy^*y = \lim_{\varepsilon \searrow 0} x \cdot (x^*x + \varepsilon)^{-1/3} \cdot (x^*x + \varepsilon)^{-1/3} \cdot x^*x \cdot (x^*x + \varepsilon)^{-1/3} = x. \qquad \square$$

Corollary 3.13. *A Hilbert B-module $\mathcal{E}$ is nondegenerate as a right B-module. The right B-module structure on $\mathcal{E}$ extends uniquely to a $\mathcal{M}(B)$-module structure. This module structure together with the given B-valued inner product, viewed as being $\mathcal{M}(B)$-valued, is a Hilbert $\mathcal{M}(B)$-module structure on $\mathcal{E}$. There are the same adjointable and compact operators on $\mathcal{E}$ as a Hilbert B- and as a Hilbert $\mathcal{M}(B)$-module.*

Proof. Proposition 3.12 implies $\mathcal{E} = \mathcal{E} \cdot B$, that is, B is a nondegenerate right module. Realise $\mathcal{E}$ as a concrete Hilbert module in $\mathbb{B}(\mathcal{H},\mathcal{K})$ by Lemma 2.9. The representation β extends to $\mathcal{M}(B)$. This is a faithful representation of $\mathcal{M}(B)$ because B is an essential ideal in it. And

$$\mathcal{E} \cdot \mathcal{M}(B) = \mathcal{E} \cdot B \cdot \mathcal{M}(B) = \mathcal{E} \cdot B = \mathcal{E}.$$

So $\mathcal{E}$ is also a concrete Hilbert $\mathcal{M}(B)$-module. Here the inner product is the same, and the right module structure is extended in the unique way to $\mathcal{M}(B)$. The definitions of the adjointable and compact operators on $\mathcal{E}$ only use the inner product. Since this is the same, the adjointable and compact operators on $\mathcal{E}$ as a Hilbert B-module or a Hilbert $\mathcal{M}(B)$-module are the same. $\square$

Exercise 3.14. *View a* C**-algebra B as a Hilbert module over itself as in Example* 1.9*. Then $\mathbb{B}(B) = \mathcal{M}(B)$ and $\mathbb{K}(B) \cong B$.*

The element y in Proposition 3.12 becomes unique under a mild extra condition:

Exercise 3.15. *For any $x \in \mathcal{E}$ there is a unique y in the closure of $x \cdot B \subseteq \mathcal{E}$ with $x = y\langle y \,|\, y\rangle_B = |y\rangle\langle y|y$.*

Theorem 3.16. *If $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$ is a concrete Hilbert B-module, then*

$$\mathbb{K}(\mathcal{E}) \cong \text{closed linear span of } \{x_1 x_2^* \in \mathbb{B}(\mathcal{K}) : x_1, x_2 \in \mathcal{E}\}.$$

If $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$ is a nondegenerate concrete Hilbert B-module, then

$$\mathbb{B}(\mathcal{E}) \cong \{T \in \mathbb{B}(\mathcal{K}) : Tx,\ T^*x \in \mathcal{E} \text{ for all } x \in \mathcal{E}\}.$$

Proof. Let $\mathcal{K}_1 \subseteq \mathcal{K}$ be the closed linear span of $x(y)$ for $x \in \mathcal{E}$, $y \in \mathcal{H}$. Let $\mathcal{K}_0$ be the orthogonal complement of $\mathcal{K}_1$. The representation of $\mathbb{K}(\mathcal{E})$ will be constructed to be 0 on $\mathcal{K}_0$. So it is equivalent to a representation on $\mathcal{K}_1$. We only consider a representation of $\mathbb{B}(\mathcal{E})$ if $\mathcal{K} = \mathcal{K}_1$. Hence we may assume without loss of generality that $\mathcal{K} = \mathcal{K}_1$. Then we identify $\mathcal{E} \otimes_B \mathcal{H} \cong \mathcal{K}$ through $x \otimes y \mapsto x(y)$.

We may view $\mathcal{E}$ as a correspondence $A = \mathbb{B}(\mathcal{E}) \xleftarrow{\mathcal{E}} B$. The proof of Lemma 2.1 gives a unital representation $\alpha\colon \mathbb{B}(\mathcal{E}) \to \mathbb{B}(\mathcal{K})$ with $\alpha(T)x(y) = (Tx)(y)$ for all $T \in \mathbb{B}(\mathcal{E})$, $x \in \mathcal{E}$, $y \in \mathcal{H}$. Then $\alpha(T)x = Tx$ in $\mathbb{B}(\mathcal{H}, \mathcal{K})$. Since $\alpha(T^*)x = T^*x$, the image $\alpha(\mathbb{B}(\mathcal{E}))$ is contained in the space of operators $T \in \mathbb{B}(\mathcal{K})$ with $Tx, T^*x \in \mathcal{E}$ for all $x \in \mathcal{E}$. Conversely, if $T \in \mathbb{B}(\mathcal{K})$ satisfies $Tx, T^*x \in \mathcal{E}$ for all $x \in \mathcal{E}$, then $x \mapsto Tx$ and $x \mapsto T^*x$ define bounded linear operators on $\mathcal{E}$, which are adjoints of each other because $x_1^*(Tx_2) = (T^*x_1)^*x_2$ for all $x_1, x_2 \in \mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$. Hence there is a unique $\hat{T} \in \mathbb{B}(\mathcal{E})$ with $\bar{\alpha}(\hat{T})x = Tx$ for all $x \in \mathcal{E}$. If $\mathcal{K} = \mathcal{K}_1$, as we assumed, then this implies $\bar{\alpha}(\hat{T}) = T$. This proves the description of $\mathbb{B}(\mathcal{E})$.

The ideal $\mathbb{K}(\mathcal{E}) \triangleleft \mathbb{B}(\mathcal{E})$ is spanned by $|x_1\rangle\langle x_2| \in \mathbb{K}(\mathcal{E})$ for $x_1, x_2 \in \mathcal{E}$. This operator acts on $\mathcal{K}$ by $\alpha\big(|x_1\rangle\langle x_2|\big)x_3(y) := \big(|x_1\rangle\langle x_2|x_3\big)(y) = x_1x_2^*x_3(y)$ for all $x_3 \in \mathcal{E}$, $y \in \mathcal{H}$. Thus $\alpha\big(|x_1\rangle\langle x_2|\big) = x_1x_2^*$. Hence the isomorphism α maps $\mathbb{K}(\mathcal{E})$ onto the closed linear span of the operators $x_1x_2^*$ for $x_1, x_2 \in \mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$. □

Corollary 3.17. $\mathcal{M}(\mathbb{K}(\mathcal{E})) = \mathbb{B}(\mathcal{E})$.

Proof. Lemma 3.9 shows that $\mathbb{K}(\mathcal{E})$ is a two-sided *-ideal in $\mathbb{B}(\mathcal{E})$. It is essential because for $T \in \mathbb{B}(\mathcal{E})$, $T = 0$ if and only if $Tx = 0$ for all $x \in \mathcal{E}$, if and only if $T|x\rangle\langle y| = 0$ for all $x, y \in \mathcal{E}$. This gives an injective *-homomorphism $\mathbb{B}(\mathcal{E}) \hookrightarrow \mathcal{M}(\mathbb{K}(\mathcal{E}))$. Realise $\mathcal{E}$ and $\mathbb{K}(\mathcal{E})$ concretely as in Theorem 3.16. We choose $\mathcal{K} = \mathcal{E} \otimes_B \mathcal{H}$, so that the representation of $\mathbb{K}(\mathcal{E})$ on $\mathcal{K}$ is nondegenerate. Then the representation of $\mathbb{K}(\mathcal{E})$ extends to a representation $\bar{\alpha}\colon \mathcal{M}(\mathbb{K}(\mathcal{E})) \to \mathbb{B}(\mathcal{K})$.

Proposition 3.12 implies $\mathcal{E} = \mathbb{K}(\mathcal{E})\mathcal{E}$. Therefore, if $T \in \mathcal{M}(\mathbb{K}(\mathcal{E}))$, then $\bar{\alpha}(T)\mathcal{E} = \bar{\alpha}(T)\mathbb{K}(\mathcal{E})\mathcal{E} \subseteq \mathbb{K}(\mathcal{E})\mathcal{E} = \mathcal{E}$ and, similarly, $\bar{\alpha}(T)^*\mathcal{E} \subseteq \mathcal{E}$. Thus $\bar{\alpha}(\mathcal{M}(\mathbb{K}(\mathcal{E})))$ is contained in the image of $\mathbb{B}(\mathcal{E})$ in $\mathbb{B}(\mathcal{K})$. Thus the canonical *-homomorphism $\mathbb{B}(\mathcal{E}) \hookrightarrow \mathcal{M}(\mathbb{K}(\mathcal{E}))$ is surjective as well. □

A *hereditary subalgebra* in a C*-algebra is a C*-subalgebra $B \subseteq A$ such that $0 \le a \le b$ with $b \in B$, $a \in A$ implies $a \in B$. This is equivalent to $BAB \subseteq B$ by [21, Theorem 3.2.2]. Always $BAB \supseteq B$ because $B \subseteq A$ and $B^3 = B$.

Theorem 3.18. *Let $\mathcal{E}$ be a Hilbert module and $\mathcal{F} \subseteq \mathcal{E}$ a Hilbert submodule. There is an injective *-homomorphism $\mathbb{K}(\mathcal{F}) \hookrightarrow \mathbb{K}(\mathcal{E})$ that maps $|x\rangle\langle y| \in \mathbb{K}(\mathcal{F})$ to $|x\rangle\langle y| \in \mathbb{K}(\mathcal{E})$. Its range is a hereditary subalgebra of $\mathbb{K}(\mathcal{E})$, and any hereditary subalgebra of $\mathbb{K}(\mathcal{E})$ is of this form for a unique Hilbert submodule $\mathcal{F} \subseteq \mathcal{E}$.*

Proof. Realise $\mathcal{E}$ as a concrete Hilbert module in $\mathbb{B}(\mathcal{H}, \mathcal{K})$ with respect to some faithful representation $\beta\colon B \hookrightarrow \mathbb{B}(\mathcal{H})$. Theorem 3.16 identifies $\mathbb{K}(\mathcal{E})$ with the C*-subalgebra of $\mathbb{B}(\mathcal{K})$ that is spanned by operators of the form $x_1x_2^*$ for $x_1, x_2 \in \mathcal{E}$. The image of $\mathcal{F} \subseteq \mathcal{E}$ in $\mathbb{B}(\mathcal{H}, \mathcal{K})$ is a concrete Hilbert B-module. Thus Theorem 3.16 identifies $\mathbb{K}(\mathcal{F})$ with the C*-subalgebra of $\mathbb{B}(\mathcal{K})$ spanned by operators of the form $x_1x_2^*$ for $x_1, x_2 \in \mathcal{F}$. These belong to the image of $\mathbb{K}(\mathcal{E})$. So we get an injective *-homomorphism $\mathbb{K}(\mathcal{F}) \hookrightarrow \mathbb{K}(\mathcal{E})$. The image satisfies $\mathbb{K}(\mathcal{F}) \cdot \mathbb{K}(\mathcal{E}) \cdot \mathbb{K}(\mathcal{F}) \subseteq \mathbb{K}(\mathcal{F})$ because

$$(x_1x_2^*)(x_3x_4^*)(x_5x_6^*) = x_1\langle x_2 \mid x_3\rangle_B (x_6\langle x_5 \mid x_4\rangle_B)^* \in \mathbb{K}(\mathcal{F})$$

for $x_1, x_2, x_5, x_6 \in \mathcal{F}$, $x_3, x_4 \in \mathcal{E}$. Thus $\mathbb{K}(\mathcal{F})$ is a hereditary subalgebra in $\mathbb{K}(\mathcal{E})$.

Conversely, let $D \subseteq \mathbb{K}(\mathcal{E})$ be hereditary. Let $\mathcal{F} := D \cdot \mathcal{E} \subseteq \mathcal{E}$. This is a Hilbert submodule. The image of $\mathbb{K}(\mathcal{F})$ in $\mathbb{K}(\mathcal{E})$ is spanned by $(d_1 x_1)(d_2 x_2)^* = d_1 x_1 x_2^* d_2^*$ for $x_1, x_2 \in \mathcal{E}$, $d_1, d_2 \in D$. These elements span $D \cdot \mathbb{K}(\mathcal{E}) \cdot D$, which is equal to D because D is hereditary. Thus $\mathbb{K}(\mathcal{F}) = D$. If $\mathcal{F} \subseteq \mathcal{E}$, then $\mathbb{K}(\mathcal{F}) \cdot \mathcal{E} = \mathcal{F}$ by Proposition 3.12. Hence the maps $\mathcal{F} \mapsto \mathbb{K}(\mathcal{F})$ and $D \mapsto D \cdot \mathcal{E}$ are inverse to each other. ☐

Example 3.19. Consider a C*-algebra B as a Hilbert module over itself. Then Hilbert submodules in B are just closed right ideals. Theorem 3.18 gives the well known bijection between right ideals $\mathcal{F}$ in B and hereditary subalgebras $D \subseteq B$. It maps $\mathcal{F} \mapsto \mathbb{K}(\mathcal{F})$, which is the closed linear span of $R \cdot R^*$, and it maps D to $D \cdot B$.

Finally, we return to the definition of a correspondence $\mathcal{E}\colon A \leftarrow B$ between two C*-algebras in Definition 2.3. The condition $\langle a\xi \mid \eta\rangle_B = \langle \xi \mid a^*\eta\rangle_B$ for the left action says that it is a *-homomorphism to the C*-algebra $\mathbb{B}(\mathcal{E})$ of adjointable operators. Corollary 3.17 identifies $\mathbb{B}(\mathcal{E}) \cong \mathcal{M}(\mathbb{K}(\mathcal{E}))$. The following lemma shows that a *-homomorphism $A \to \mathbb{B}(\mathcal{E})$ satisfies $A \cdot \mathcal{E} = \mathcal{E}$ if and only if it is nondegenerate as a *-homomorphism to $\mathcal{M}(\mathbb{K}(\mathcal{E}))$:

Lemma 3.20. *Let $\mathcal{E}$ be a Hilbert B-module, $\varphi\colon A \to \mathbb{B}(\mathcal{E})$ a *-homomorphism. Then $A \cdot \mathcal{E} = \mathcal{E}$ if and only if $A \cdot \mathbb{K}(\mathcal{E}) = \mathbb{K}(\mathcal{E})$. So a correspondence $A \leftarrow B$ is the same as a Hilbert B-module $\mathcal{E}$ with a morphism $A \to \mathbb{K}(\mathcal{E})$.*

Proof. If $A \cdot \mathcal{E} = \mathcal{E}$, then the operators of the form $a|x\rangle\langle y| = |ax\rangle\langle y|$ also span a dense subspace in $\mathbb{K}(\mathcal{E})$, so $A \cdot \mathbb{K}(\mathcal{E}) = \mathbb{K}(\mathcal{E})$. Conversely, by Proposition 3.12, any element of $\mathcal{E}$ has the form $|y\rangle\langle y|y = y\langle y \mid y\rangle$. Thus $\mathcal{E}$ is a nondegenerate $\mathbb{K}(\mathcal{E})$-module. Hence $A \cdot \mathbb{K}(\mathcal{E}) = \mathbb{K}(\mathcal{E})$ implies $A \cdot \mathcal{E} = A \cdot \mathbb{K}(\mathcal{E})\mathcal{E} = \mathbb{K}(\mathcal{E})\mathcal{E} = \mathcal{E}$. ☐

4. Composition of correspondences

Let $\mathcal{E}\colon A \leftarrow B$ and $\mathcal{F}\colon B \leftarrow C$ be correspondences. The algebraic tensor product $\mathcal{E} \odot_B \mathcal{F}$ is an A,C-bimodule via $a(x \otimes y)c = ax \otimes yc$. We define a C-valued inner product on $\mathcal{E} \odot_B \mathcal{F}$ by

$$\langle x_1 \otimes y_1 \mid x_2 \otimes y_2\rangle = \langle y_1 \mid \langle x_1 \mid x_2\rangle_B y_2\rangle_C \tag{4.1}$$

for $x_1, x_2 \in \mathcal{E}$, $y_1, y_2 \in \mathcal{F}$, exactly as in (1.2). We compute

$$\langle x_1 \otimes y_1 \mid x_2 b \otimes y_2\rangle = \langle x_1 \otimes y_1 \mid x_2 \otimes b y_2\rangle$$

for all $x_1, x_2 \in \mathcal{E}$, $b \in B$, $y_1, y_2 \in \mathcal{F}$, and similarly in the first variable. Thus the inner product descends to the balanced tensor product $\mathcal{E} \odot_B \mathcal{F}$.

Lemma 4.1. *Equation* (4.1) *defines a positive semidefinite inner product on $\mathcal{E} \odot_B \mathcal{F}$. Thus the completion in this inner product gives a Hilbert C-module, which we denote by $\mathcal{E} \otimes_B \mathcal{F}$. There is a unique nondegenerate *-homomorphism $A \to \mathbb{B}(\mathcal{E} \otimes_B \mathcal{F})$ with $a \cdot (x \otimes y) = (ax) \otimes y$ for all $a \in A$, $x \in \mathcal{E}$, $y \in \mathcal{F}$.*

The lemma follows immediately when we realise all correspondences concretely:

Proposition 4.2. *Let A, B, C be* C*-*algebras, let $\mathcal{E}\colon A \leftarrow B$, $\mathcal{F}\colon B \leftarrow C$ be correspondences, let $\mathcal{H}$ be a Hilbert space, and let $\gamma\colon C \hookrightarrow \mathbb{B}(\mathcal{H})$ be a faithful representation. There are Hilbert spaces $\mathcal{K}$ and $\mathcal{L}$ with faithful representations*

$$B \overset{\beta}{\hookrightarrow} \mathbb{B}(\mathcal{K}), \quad A \overset{\alpha}{\hookrightarrow} \mathbb{B}(\mathcal{L}), \quad \mathcal{E} \overset{\varepsilon}{\hookrightarrow} \mathbb{B}(\mathcal{K},\mathcal{L}), \quad \mathcal{F} \overset{\varphi}{\hookrightarrow} \mathbb{B}(\mathcal{H},\mathcal{K})$$

such that

$$\alpha(a)\varepsilon(x)\beta(b) = \varepsilon(axb), \qquad \varepsilon(x_1)^*\varepsilon(x_2) = \beta(\langle x_1 \mid x_2\rangle_B),$$
$$\beta(b)\varphi(y)\gamma(c) = \varphi(byc), \qquad \varphi(y_1)^*\varphi(x_2) = \gamma(\langle y_1 \mid y_2\rangle_C)$$

for all $a \in A$, $x, x_1, x_2 \in \mathcal{E}$, $b \in B$, $y, y_1, y_2 \in \mathcal{F}$, $c \in C$. Given maps $\alpha, \beta, \gamma, \varphi, \varepsilon$ as above – but not necessarily faithful – the map

$$\psi\colon \mathcal{E} \odot_B \mathcal{F} \to \mathbb{B}(\mathcal{H}, \mathcal{L}), \qquad x \otimes y \mapsto \varepsilon(x)\varphi(y),$$

extends to an isometric A, C-bimodule isomorphism from $\mathcal{E} \odot_B \mathcal{F}$ onto a subspace of $\mathbb{B}(\mathcal{H}, \mathcal{L})$, which completes to a concrete correspondence $A \leftarrow C$.

Proof. By Lemma 2.9, there is an isomorphism $\varphi_0\colon \mathcal{F} \hookrightarrow \mathbb{B}(\mathcal{H}, \mathcal{K}_0)$ from $\mathcal{F}$ onto a concrete correspondence $B \leftarrow C$ with respect to a faithful representation $\beta_0\colon B \hookrightarrow \mathbb{B}(\mathcal{K}_0)$ and the given faithful representation $\gamma\colon C \hookrightarrow \mathbb{B}(\mathcal{H})$. Applying Lemma 2.9 to $\mathcal{E}$ and the homomorphism β gives a concrete realisation $\varepsilon\colon \mathcal{E} \hookrightarrow \mathbb{B}(\mathcal{K}, \mathcal{L})$ of the correspondence $\mathcal{E}$ with a faithful representation $\alpha\colon A \hookrightarrow \mathbb{B}(\mathcal{L})$.

Next, we compute

$$\begin{aligned}
\alpha(a)\psi(x \otimes y)\gamma(c) &= \alpha(a)\varepsilon(x)\varphi(y)\gamma(c) = \varepsilon(ax)\varphi(yc) = \psi(ax \otimes yc),\\
\psi(x_1 \otimes y_1)^*\psi(x_2 \otimes y_2) &= \varphi(y_1)^*\varepsilon(x_1)^*\varepsilon(x_2)\varphi(y_2) = \varphi(y_1)^*\beta(\langle x_1 \mid x_2\rangle_B)\varphi(y_2)\\
&= \varphi(y_1)^*\varphi(\langle x_1 \mid x_2\rangle_B y_2) = \langle y_1 \mid \langle x_1 \mid x_2\rangle_B \cdot y_2\rangle_C.
\end{aligned}$$

Thus ψ extends to an isometric linear map $\mathcal{E} \odot_B \mathcal{F} \hookrightarrow \mathbb{B}(\mathcal{H}, \mathcal{L})$. The norm closure of $\mathcal{E} \odot_B \mathcal{F}$ is a concrete correspondence $A \leftarrow C$. This implies Lemma 4.1. □

Example 4.3. Correspondences $B \leftarrow \mathbb{C}$ are the same as Hilbert space representations of B (see Example 1.8). The functor between the Hilbert space representations of B and A induced by a correspondence that we used to motivate the definition of correspondences is a special case of the composition of correspondences. Similarly, composition with a correspondence $A \leftarrow B$ gives a functor from the category of Hilbert A-modules to the category of Hilbert B-modules.

As with bimodules between rings, the composition of correspondences is only associative and unital up to canonical isomorphisms:

Lemma 4.4. *Let $\mathcal{E}\colon A \leftarrow B$, $\mathcal{F}\colon B \leftarrow C$, and $\mathcal{G}\colon C \leftarrow D$ be correspondences. Then there are natural isomorphisms of correspondences*

$$\begin{aligned}
(\mathcal{E} \otimes_B \mathcal{F}) \otimes_C \mathcal{G} &\cong \mathcal{E} \otimes_B (\mathcal{F} \otimes_C \mathcal{G}), &\qquad (x \otimes y) \otimes z &\mapsto x \otimes (y \otimes z),\\
\mathcal{E} \otimes_B B &\cong \mathcal{E}, & x \otimes b &\mapsto x \cdot b,\\
A \otimes_A \mathcal{E} &\cong \mathcal{E}, & a \otimes x &\mapsto a \cdot x.
\end{aligned}$$

Proof. The three maps above are defined on vectors that span the respective Hilbert modules. Direct computations show that they are bimodule maps and preserve the inner products. Hence they extend to isometric linear maps on the completions. Since these extensions have dense range, they are unitary. □

4.1. **The correspondence bicategory.** We have now learnt enough about correspondences to turn them into a bicategory. Its objects are C*-algebras, arrows $B \to A$ are correspondences $\mathcal{E}\colon A \leftarrow B$, and 2-arrows are correspondence maps, that is, A, B-bimodule maps $i\colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ with $\langle i(x) \mid i(y)\rangle_B = \langle x \mid y\rangle_B$ for all $x, y \in \mathcal{E}_1$. The *composition* of arrows is the tensor product $\otimes_B$. The *unit arrow* on A is the *identity correspondence*: A with the usual bimodule structure and the inner product $\langle x \mid y\rangle_A = x^*y$ for $x, y \in A$ (see Example 1.9). The *vertical composition* of 2-arrows is the composition of correspondence maps $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \mathcal{E}_3$. This is associative, and the identity map on a correspondence is a unit 2-arrow for the vertical composition.

Definition 4.5. Let A and B be C*-algebras. Let $\mathfrak{Corr}(A, B)$ be the category with correspondences $A \leftarrow B$ as objects and correspondence maps as arrows.

Let $\mathcal{E}_1, \mathcal{E}_2 \colon A \leftarrow B$ and $\mathcal{F}_1, \mathcal{F}_2 \colon B \leftarrow C$ be pairs of parallel composable correspondences. Two correspondence maps $i \colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ and $j \colon \mathcal{F} \hookrightarrow \mathcal{F}_2$ induce an isometric bimodule map $i \odot_B j \colon \mathcal{E}_1 \odot_B \mathcal{F} \hookrightarrow \mathcal{E}_2 \odot_B \mathcal{F}_2$, $x \otimes y \mapsto i(x) \otimes j(y)$. This extends to a correspondence map $i \otimes_B j \colon \mathcal{E}_1 \otimes_B \mathcal{F} \hookrightarrow \mathcal{E}_2 \otimes_B \mathcal{F}_2$. This defines the *horizontal composition* of i and j. The tensor products for correspondences and correspondence maps define a bifunctor

$$\otimes_B \colon \mathfrak{Corr}(A, B) \times \mathfrak{Corr}(B, C) \to \mathfrak{Corr}(A, C),$$

that is, the horizontal and vertical compositions of 2-arrows commute and preserve unit 2-arrows. Lemma 4.4 gives invertible 2-arrows

$$l_{\mathcal{E}} \colon A \otimes_A \mathcal{E} \cong \mathcal{E}, \qquad r_{\mathcal{E}} \colon \mathcal{E} \otimes_B B \cong \mathcal{E}, \qquad \mathrm{ass} \colon (\mathcal{E} \otimes_B \mathcal{F} \otimes_C)\mathcal{G} \cong \mathcal{E} \otimes_B (\mathcal{F} \otimes_C \mathcal{G})$$

in $\mathfrak{Corr}$. They are natural with respect to correspondence maps because they are defined in terms of inner products and bimodule structures. Direct computations, which are the same as for $\mathfrak{Rings}$, show that the relevant diagrams commute. We have shown:

Theorem 4.6. $\mathfrak{Corr}$ *is a bicategory.*

The isomorphisms of correspondences are exactly the invertible 2-arrows in $\mathfrak{Corr}$, and restricting to invertible 2-arrows in a bicategory always gives another bicategory. Previous articles on the correspondence bicategory only deal with this subbicategory of $\mathfrak{Corr}$. There are, however, some applications that require the non-invertible 2-arrows introduced here. One involves representations of *-algebras by unbounded operators in [19]. The other, to Fell bundles over inverse semigroups, is explained below in Section 7.

Remark 4.7. Why do we require the left action in a correspondence to be nondegenerate? Otherwise, the left unit for composition would no longer work: $A \otimes_A \mathcal{E} \cong A \cdot \mathcal{E} \neq \mathcal{E}$ if the left A-action on $\mathcal{E}$ is degenerate. Assuming nondegeneracy for correspondences is almost no restriction: if $A \to \mathbb{B}(\mathcal{E})$ is degenerate, then we may replace $\mathcal{E}$ by the Hilbert submodule $A \cdot \mathcal{E} \subseteq \mathcal{E}$; this with the restriction of the left action of A becomes a correspondence.

Remark 4.8. The isomorphism classes of arrows in a bicategory always form a category. This category shadow of $\mathfrak{Corr}$ is considered by Echterhoff, Kaliszewski, Quigg and Raeburn in [8].

Landsman [16] has introduced several bicategories to unify the concepts of Morita equivalence for rings, C*-algebras, von Neumann algebras, Lie groupoids, symplectic groupoids, and integrable Poisson manifolds. He calls C*-correspondences "Hilbert bimodules", and his bicategory differs from ours in having all adjointable bimodule maps as 2-arrows.

4.2. **Proper, full and faithful correspondences.**

Definition 4.9. Let A and B be C*-algebras. A correspondence $\mathcal{E} \colon A \leftarrow B$ is *full* if $\langle \mathcal{E} \,|\, \mathcal{E} \rangle_B$, the closed two-sided ideal generated by the inner products $\langle \xi \,|\, \eta \rangle$ with $\xi, \eta \in \mathcal{E}$, is equal to B; it is *proper* if the left action is a *-homomorphism $A \to \mathbb{K}(\mathcal{E})$, and *faithful* if the left action is an injective *-homomorphism $A \to \mathbb{B}(\mathcal{E})$.

For any Hilbert B-module $\mathcal{E}$, the closed linear span $I := \langle \mathcal{E} \,|\, \mathcal{E} \rangle_B$ is a closed *-ideal in B. Restricting the right module structure on $\mathcal{E}$ to I, $\mathcal{E}$ becomes a full Hilbert I-module. Conversely, any full Hilbert I-module for an ideal $I \triangleleft B$ is of this form for a unique Hilbert B-module structure on $\mathcal{E}$ by Corollary 3.13.

Lemma 4.10. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $\beta \colon B \hookrightarrow \mathbb{B}(\mathcal{H})$ and $\alpha \colon A \to \mathbb{B}(\mathcal{K})$ be representations, with β faithful. Let $\mathcal{E} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$ be a nondegenerate* *concrete correspondence* $A \overset{C}{\leftarrow} B$*. Then* $\mathcal{E}$ *is full if and only if* $\mathcal{E}^* \cdot \mathcal{E} = \beta(B)$*; proper if and only if* $\alpha(A) \subseteq \mathcal{E}\mathcal{E}^*$*; and faithful if and only if* α *is faithful.*

Here $\mathcal{E}^* \cdot \mathcal{E}$ and $\mathcal{E} \cdot \mathcal{E}^*$ denote the closed linear spans of products x^*y or yx^* for $x, y \in \mathcal{E}$, respectively.

Proof. The B-valued inner product on $\mathcal{E}$ is $\langle x \mid y \rangle = \beta^{-1}(x^*y)$. So $\mathcal{E}^* \cdot \mathcal{E} = \beta(B)$ if and only $\mathcal{E}$ is full. Theorem 3.16 realises $\mathbb{K}(\mathcal{E})$ faithfully on $\mathcal{K}$. Since $x(y)$ for $x \in \mathcal{E}$, $y \in \mathcal{H}$ generate $\mathcal{K}$, an operator T on $\mathcal{K}$ is determined by its composites Tx for $x \in \mathcal{E}$. Hence $\mathcal{E}$ is proper if and only if the image of A in $\mathbb{B}(\mathcal{K})$ is contained in the image of $\mathbb{K}(\mathcal{E})$ in $\mathbb{B}(\mathcal{K})$. By Theorem 3.16, this means that $\alpha(A) \subseteq \mathcal{E} \cdot \mathcal{E}^*$.

Let α be faithful and let $a \in A$. If $a\mathcal{E} = 0$, then $\alpha(a)x = 0$ for all $x \in \mathcal{E}$. Hence $\alpha(a)x(y) = 0$ for all $x \in \mathcal{E}$, $y \in \mathcal{H}$. This implies $\alpha(a) = 0$ and hence $a = 0$. Conversely, if α is not faithful, then there is $a \in A$ with $a \neq 0$ and $\alpha(a) = 0$. Hence $\alpha(a)x = a \cdot x = 0$ for all $x \in \mathcal{E}$, showing that $a \cdot \mathcal{E} = 0$ because $\mathcal{E} \to \mathbb{B}(\mathcal{H}, \mathcal{K})$ is faithful. $\square$

Proposition 4.11. *Let* $\mathcal{E}\colon A \overset{C}{\leftarrow} B$ *and* $\mathcal{F}\colon B \overset{C}{\leftarrow} C$ *be correspondences. If both* $\mathcal{E}$ *and* $\mathcal{F}$ *are proper, full or faithful, then so is* $\mathcal{E} \otimes_B \mathcal{F}$*.*

Proof. We describe $\mathcal{E} \otimes_B \mathcal{F}$ concretely as in Proposition 4.2, and use the notation there. If $\mathcal{E}$ and $\mathcal{F}$ are full, then $\varepsilon(\mathcal{E})^*\varepsilon(\mathcal{E}) = \beta(B)$ and $\varphi(\mathcal{F})^*\varphi(\mathcal{F}) = \gamma(C)$ by Lemma 4.10. And $\beta(B)\varphi(\mathcal{F}) = \varphi(\mathcal{F})$ because $B \cdot \mathcal{F} = \mathcal{F}$. Hence

$$\begin{aligned}\psi(\mathcal{E} \otimes_B \mathcal{F})^*\psi(\mathcal{E} \otimes_B \mathcal{F}) &= \varphi(\mathcal{F})^*\varepsilon(\mathcal{E})^*\varepsilon(\mathcal{E})\varphi(\mathcal{F}) \\ &= \varphi(\mathcal{F})^*\beta(B)\varphi(\mathcal{F}) = \varphi(\mathcal{F})^*\varphi(\mathcal{F}) = \gamma(C).\end{aligned}$$

Thus $\mathcal{E} \otimes_B \mathcal{F}$ is full by Lemma 4.10.

If $\mathcal{E}$ and $\mathcal{F}$ are proper, then $\varepsilon(\mathcal{E})\varepsilon(\mathcal{E})^* \supseteq \alpha(A)$ and $\varphi(\mathcal{F})\varphi(\mathcal{F})^* \supseteq \beta(B)$ by Lemma 4.10, and $\varepsilon(\mathcal{E})\beta(B) = \varepsilon(\mathcal{E})$ because $\mathcal{E} \cdot B = \mathcal{E}$. Hence

$$\begin{aligned}\psi(\mathcal{E} \otimes_B \mathcal{F})\psi(\mathcal{E} \otimes_B \mathcal{F})^* &= \varepsilon(\mathcal{E})\varphi(\mathcal{F})\varphi(\mathcal{F})^*\varepsilon(\mathcal{E})^* \\ &\supseteq \varepsilon(\mathcal{E})\beta(B)\varepsilon(\mathcal{E})^* = \varepsilon(\mathcal{E})\varepsilon(\mathcal{E})^* \supseteq \alpha(A).\end{aligned}$$

Thus $\mathcal{E} \otimes_B \mathcal{F}$ is proper by Lemma 4.10.

If $\mathcal{F}$ and $\mathcal{E}$ are faithful, then the representations β and α are faithful by Lemma 4.10. Then $\mathcal{E} \otimes_B \mathcal{F}$ is faithful. $\square$

Thus the full, the proper and the faithful correspondences form subbicategories in $\mathfrak{Corr}$, denoted $\mathfrak{Corr}^{\mathrm{full}}$ and $\mathfrak{Corr}^{\mathrm{prop}}$, $\mathfrak{Corr}^{\mathrm{faith}}$, respectively. Intersections such as $\mathfrak{Corr}^{\mathrm{full}} \cap \mathfrak{Corr}^{\mathrm{prop}}$ are also subbicategories of $\mathfrak{Corr}$.

One reason why proper correspondences are important is that they induce maps on K-theory, even in bivariant KK. A second reason are the results on Cuntz–Pimsner algebras of single endomorphisms and product systems in [1,2], which only work for proper correspondences.

4.3. **Morphisms and *-homomorphisms as correspondences.** A morphism $f\colon A \to B$ yields a correspondence B_f by Example 2.11. What are the 2-arrows $B_f \Rightarrow B_g$ in $\mathfrak{Corr}$? We shall first study invertible 2-arrows. Since $\mathbb{B}(B) = \mathcal{M}(B)$ by Exercise 3.14, an isomorphism of right Hilbert B-modules $B_f \cong B_g$ is just a unitary multiplier of B. A unitary $u \in \mathcal{U}(B)$ gives an isomorphism of correspondences $B_f \cong B_g$ if and only if $u(f(a)b) = g(a)u(b)$ for all $a \in A$, $b \in B$. Thus isomorphisms of correspondences $B_f \xrightarrow{\cong} B_g$ are the same as 2-arrows $f \Rightarrow g$ in $\mathcal{C}^*(2)$.

Of course, the vertical composition of 2-arrows is also the same. The following lemma clarifies how the compositions of arrows in $\mathcal{C}^*(2)$ and $\mathfrak{Corr}$ are related.

Lemma 4.12. *Let $f\colon A \to \mathcal{M}(B)$ and $g\colon B \to \mathcal{M}(C)$ be morphisms. Then*

$$\mu_{f,g}\colon B_f \otimes_B C_g \xrightarrow{\cong} C_{g\circ f}, \qquad b \otimes c \mapsto g(b)\cdot c,$$

is an isomorphism of correspondences. If $u\colon f_1 \Rightarrow f_2$ and $v\colon g_1 \Rightarrow g_2$ are 2-arrows in $\mathcal{C}^*(2)$, then the following diagram of correspondence isomorphisms commutes:

$$\begin{tikzcd}
B_{f_1} \otimes_B C_{g_1} \arrow[r, Rightarrow, "\mu_{f_1,g_1}"] \arrow[d, Rightarrow, "u\otimes_B v"'] & C_{g_1\circ f_1} \arrow[d, Rightarrow, "v\bullet u"] \\
B_{f_2} \otimes_B C_{g_2} \arrow[r, Rightarrow, "\mu_{f_2,g_2}"] & C_{g_2\circ f_2}
\end{tikzcd}$$

Proof. The map $\mu_{f,g}$ is an isometry of right Hilbert C-modules. Its range is dense because g is nondegenerate. It is left A-linear: $f(a)\cdot b\otimes c \mapsto g(f(a)\cdot b)\cdot c = g\circ f(a)\cdot(g(b)\cdot c)$. Thus it is an isomorphism of correspondences. The commuting diagram is trivial to check. $\square$

The identity map on B gives the unit correspondence on B. Thus the identity map on objects, the map $f \mapsto B_f$ on arrows, the map $u \mapsto u$ on 2-arrows, the multiplication maps $\mu_{f,g}$ above, and the identity map $\lambda\colon B_{\mathrm{id}_B} \to B$ provide a strictly unital homomorphism from the opposite of the bicategory $\mathcal{C}^*(2)$ to $\mathfrak{Corr}$. The order of multiplication is reversed because of our convention that an A,B-correspondence is an arrow from B to A.

The correspondences of the form B_f for morphisms f do not quite form a subbicategory because the composite $B_f \otimes_B C_g$ is only *isomorphic to* $C_{g\circ f}$. A correspondence $\mathcal{E}\colon A \xleftarrow{c} B$ is isomorphic to one of the form B_f if and only if $\mathcal{E} \cong B$ as a Hilbert B-module. By Lemma 4.12, there is a subbicategory of $\mathfrak{Corr}$ that has correspondences of this form as arrows and isomorphisms of correspondences as 2-arrows. The homomorphism $\mathcal{C}^*(2)^{\mathrm{op}} \to \mathfrak{Corr}'$ is surjective on objects and, for each pair of C*-algebras A, B, the functor $\mathcal{C}^*(2)(A,B) \to \mathfrak{Corr}'(A,B)$ is an equivalence of categories; here we use that an isomorphism of correspondences $B_f \xrightarrow{\cong} B_g$ is the same as a unitary in $\mathcal{M}(B) = \mathbb{B}(B)$ that intertwines f and g. Hence the functor $\mathcal{C}^*(2)^{\mathrm{op}} \to \mathfrak{Corr}'$ is an equivalence of bicategories.

The correspondence B_f associated to a morphism $f\colon A \to\!\!\!\!\!\!\rightarrow B$ is always full. It is proper or faithful if and only if the morphism f is proper or faithful, respectively. Here we use that $\mathbb{K}(A) = A$ if A is viewed as a right Hilbert A-module.

The bicategory $\mathfrak{Corr}$ also has non-invertible 2-arrows $B_f \Rightarrow B_g$. A 2-arrow $i\colon B_f \hookrightarrow B_g$ is a linear map $i\colon B \to B$ such that $i(b_1)b_2 = i(b_1b_2)$ and $i(b_1)^*i(b_2) = b_1^*b_2$ for all $b_1, b_2 \in B$ and $i(f(a)b) = g(a)i(b)$ for all $a \in A$, $b \in B$. The image $i(B) \subseteq B$ is a right ideal in B. If $i(B) = B$, then i is a unitary intertwiner between f and g. If B is unital, then $i(1)$ is a projection in $\mathcal{M}(B)$ with $i(B) = i(1)\cdot B$, so $i(B)$ is automatically complementable. Hence the following lemma always applies if B is unital:

Lemma 4.13. *The 2-arrows $B_f \Rightarrow B_g$ in $\mathfrak{Corr}$ where the right ideal $i(B) \subseteq B$ is complementable are in bijection with isometries $i \in \mathcal{M}(B)$ such that $g(A)$ commutes with the range projection ii^* and $f(a) = i^*g(a)i$ for all $a \in A$.*

Proof. If the right ideal $i(B)$ is complementable, then i is adjointable, its adjoint being the orthogonal projection to $i(B)$. Conversely, if i is adjointable, then i is an isometry in $\mathcal{M}(B)$, and ii^* is an orthogonal projection onto $i(B)$, so that $i(B)$ is complementable.

The intertwining condition says that $if(a) = g(a)i$ for all $a \in A$. Hence $i^*if(a) = i^*g(a)i$ and $g(a)ii^* = if(a)i^* = ii^*if(a)i^* = ii^*g(a)ii^*$ for all $a \in A$. The second equation for a^* gives $ii^*g(a) = ii^*g(a)ii^*$ as well. So the projection ii^* commutes with $g(A)$. Conversely, if $i \in \mathcal{M}(B)$ is an isometry whose range projection ii^*

commutes with $g(A)$, then $f(a) := i^* g(a) i$ is a morphism $f\colon A \rightsquigarrow B$. And i gives a 2-arrow $i\colon B_f \Rightarrow B_g$ with complementable image. □

Example 4.14. Let $f\colon A \to B$ be a *-homomorphism, possibly degenerate. We may still construct a correspondence: let $B_f := f(A)B \subseteq B$. This is a right ideal in B and hence a Hilbert B-submodule of B. The left A-action defined using f is nondegenerate on this submodule. Hence B_f is a correspondence $A \xleftarrow{\mathcal{C}} B$.

Exercise 4.15. *The correspondence B_f for a *-homomorphism $f\colon A \to B$ is always proper. It is full if and only if $f(A) \subseteq B$ is not contained in any proper ideal in B. A correspondence $\mathcal{E}\colon A \xleftarrow{\mathcal{C}} B$ is isomorphic to B_f for some *-homomorphism $f\colon A \to B$ if and only if $\mathcal{E}$ is proper and there is an isometric embedding of Hilbert B-modules $\mathcal{E} \hookrightarrow B$.*

We may also view a possibly degenerate *-homomorphism $A \to B$ as a proper morphism $A \to B'$ for a hereditary subalgebra $B' \subseteq B$. Then $B' \cdot B \subseteq B$ is a right ideal and $B' = \mathbb{K}(B' \cdot B)$ (see Example 3.19). Thus the proper morphism $A \to B'$ leads to the same correspondence as in Example 4.14.

Lemma 4.16. *Let $f\colon A \to B$ and $g\colon B \to C$ be *-homomorphisms. Then there is an isomorphism of correspondences*

$$B_f \otimes_B C_g \to C_{g\circ f}, \qquad b \otimes c \mapsto g(b)c.$$

Proof. The map above is isometric and compatible with the left A-module structure. If $b \in f(A)B$, $c \in g(B)C$, then $g(b)c \in gf(A)g(B)C$. We have $gf(A)g(B)C \subseteq gf(A)C$ because $g(B)C \subseteq C$ and $gf(A)g(B)C \supseteq gf(A)C$ because $gf(A)g(B) \supseteq g(f(A))g(f(A)) \supseteq gf(A)$. Hence the map $B_f \otimes_B C_g \to C_{g\circ f}$ is also surjective. Then it is unitary. □

As for $\mathcal{C}^*(2)$, Lemma 4.16 shows that $f \mapsto B_f$ is part of a homomorphism from the opposite of the category $\mathcal{C}^*_+$, viewed as a bicategory, to $\mathfrak{Corr}$. Working in $\mathfrak{Corr}$ gives 2-categorical enrichments of $\mathcal{C}^*_+$, where the 2-arrows $f \Rightarrow g$ for $f, g\colon A \rightrightarrows B$ are all or merely the invertible 2-arrows $B_f \Rightarrow B_g$ in $\mathfrak{Corr}$. These 2-arrows seem unnatural, however, unless we work in the setting of C*-correspondences.

5. Sums and tensor products of correspondences

Let $(\mathcal{E}_i)_{i\in I}$ be a set of Hilbert B-modules. We define their *orthogonal direct sum* $\bigoplus_{i\in I} \mathcal{E}_i$ as the completion of the algebraic direct sum with respect to the B-valued inner product

$$\langle (x_i)_{i\in I} \mid (y_i)_{i\in I} \rangle_B := \sum_{i\in I} \langle x_i \mid y_i \rangle_B.$$

This inner product is again positive definite and hence turns the algebraic direct sum into a pre-Hilbert B-module. A special case of this are the Hilbert B-modules B^n for $n \in \mathbb{N} \cup \{\infty\}$ in Example 1.10 and the Hilbert B-modules $\mathcal{E}^n$ in Example 1.11.

Let $(A_i)_{i\in I}$ be C*-algebras. Define $\bigoplus_{i\in I} A_i$ to be the C_0-direct sum, consisting of all families $(a_i)_{i\in I} \in \prod_{i\in I} A_i$ with $\|a_i\|_{i\in I} \in \mathrm{C}_0(I)$.

Proposition 5.1. *Let A_i for $i \in I$ and B be* C*-*algebras. Then there are equivalences of categories*

$$\mathfrak{Corr}\Big(\bigoplus_{i\in I} A_i, B\Big) \cong \prod_{i\in I} \mathfrak{Corr}(A_i, B),$$

$$\mathfrak{Corr}\Big(B, \bigoplus_{i\in I} A_i\Big) \cong \prod_{i\in I} \mathfrak{Corr}(B, A_i).$$

That is, $\bigoplus_{i\in I} A_i$ is both a product and a coproduct of the set of objects $(A_i)_{i\in I}$ in the correspondence bicategory.

Proof. Given correspondences $\mathcal{E}_i\colon A_i \leftarrow B$, we may form the Hilbert B-module $\bigoplus_{i\in I} \mathcal{E}_i$ and equip it with a nondegenerate left action of $\bigoplus_{i\in I} A_i$ to get a correspondence $\bigoplus_{i\in I} A_i \leftarrow B$. Embeddings of correspondences $\mathcal{E}_i \hookrightarrow \mathcal{E}'_i$ may be put together to an embedding of correspondences $\bigoplus_{i\in I} \mathcal{E}_i \hookrightarrow \bigoplus_{i\in I} \mathcal{E}'_i$. This defines a functor

$$\prod_{i\in I} \mathfrak{Corr}(A_i, B) \to \mathfrak{Corr}\Bigl(\bigoplus_{i\in I} A_i, B\Bigr).$$

We are going to prove that the functor above is essentially surjective. Let $\mathcal{E}\colon \bigoplus_{i\in I} A_i \leftarrow B$ be a correspondence. Since the left action is nondegenerate, it extends to an action of the multiplier algebra. A multiplier of $\bigoplus_{i\in I} A_i$ restricts to a multiplier of A_i for each $i \in I$. This defines a map from the multiplier algebra to the C*-algebraic product $\prod_{i\in I} \mathcal{M}(A_i)$, which consists of all uniformly bounded families $a_i \in \mathcal{M}(A_i)$. This map is easily seen to be an isomorphism. In particular, $\mathcal{M}\bigl(\bigoplus_{i\in I} A_i\bigr)$ contains orthogonal projections p_i onto the ith summand with strict convergence $\sum_{i\in I} p_i = 1$. These act by orthogonal projections on $\mathcal{E}$. Let $\mathcal{E}_i := p_i\mathcal{E}$ be their images; these are Hilbert submodules on which A_i acts nondegenerately, respectively. Thus $\mathcal{E}_i\colon A_i \leftarrow B$. And $\sum_{i\in I} p_i = 1$ implies $\sum_{i\in I} \mathcal{E}_i = \mathcal{E}$. Thus $\mathcal{E}$ belongs to the essential range of our functor.

Any $\bigoplus A_i, B$-correspondence map $\bigoplus \mathcal{E}_i \hookrightarrow \bigoplus \mathcal{E}'_i$ commutes with the left action of the multiplier algebra and hence with the projections p_i. So it comes from a family of correspondence maps on the summands $\mathcal{E}_i$; that is, our functor is fully faithful. Hence it is an equivalence of categories.

Now consider a family of correspondences $\mathcal{E}_i\colon B \leftarrow A_i$. Let $\bigoplus_{i\in I} \mathcal{E}_i$ be the set of all families $(\xi_i)_{i\in I}$ with $\|\xi_i\| \in \mathrm{C}_0(I)$. This is a Hilbert module over $\bigoplus_{i\in I} A_i$ by the pointwise operations. The left actions of B on the summands $\mathcal{E}_i$ give a nondegenerate left action of B on $\bigoplus_{i\in I} \mathcal{E}_i$. Thus we get a correspondence $B \leftarrow \bigoplus_{i\in I} A_i$. This construction is natural for embeddings of correspondences, that is, there is a functor

$$\prod_{i\in I} \mathfrak{Corr}(B, A_i) \to \mathfrak{Corr}\Bigl(B, \bigoplus_{i\in I} A_i\Bigr).$$

Take $\mathcal{E}\colon B \leftarrow \bigoplus_{i\in I} A_i$. For each $i \in I$, $\mathcal{E}_i := \mathcal{E} \cdot A_i \subseteq \mathcal{E}$ is a B, A_i-correspondence for the ideal $A_i \triangleleft \bigoplus_{j\in I} A_j$. Since these ideals are orthogonal, $\mathcal{E} \cong \bigoplus_{i\in I} \mathcal{E}_i$. Thus $\mathcal{E}$ belongs to the essential range of our functor. Since the above decomposition is natural, our functor is fully faithful. □

Now let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert modules over different C*-algebras B and C, respectively. We are going to define an exterior product Hilbert module $\mathcal{E} \otimes \mathcal{F}$ over $B \otimes C$, where $B \otimes C$ is a C*-tensor product of B and C. This should be the completion of the algebraic tensor product $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$, equipped with the obvious right $B \otimes_{\mathrm{alg}} C$-module structure and the $B \otimes_{\mathrm{alg}} C$-valued inner product defined by

$$\langle x_1 \otimes y_1 \,|\, x_2 \otimes y_2\rangle := \langle x_1 \,|\, x_2\rangle_B \otimes \langle y_1 \,|\, y_2\rangle_C$$

for $x_1, x_2 \in \mathcal{E}$, $y_1, y_2 \in \mathcal{F}$. This is $\mathbb{C}$- and $B \otimes_{\mathrm{alg}} C$-linear in the second variable and satisfies $\langle x_2 \otimes y_2 \,|\, x_1 \otimes y_1\rangle = \langle x_1 \otimes y_1 \,|\, x_2 \otimes y_2\rangle^*$.

Lemma 5.2. *The inner product above is positive definite, that is, $\langle \xi \,|\, \xi\rangle \geq 0$ in $B \otimes C$ for all $\xi \in \mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$. If the norm on $B \otimes C$ is a cross norm, then the inner product satisfies $\langle \xi \,|\, \xi\rangle \neq 0$ for $\xi \neq 0$.*

Proof. Let $\pi\colon B \otimes C \hookrightarrow \mathbb{B}(\mathcal{H})$ be a faithful Hilbert space representation. This representation is of the form $\pi(b \otimes c) = \pi_B(b) \cdot \pi_C(c)$, where π_B and π_C are representations of B and C on $\mathcal{H}$ with commuting ranges. Let $\mathcal{K} := \mathcal{E} \otimes_{\pi_B} \mathcal{H}$ and represent $\mathcal{E}$ by operators $\mathcal{H} \to \mathcal{K}$ as in Lemma 1.17.

If $u \in \mathcal{M}(C)$ is unitary, then $\pi_C(u)$ is an automorphism of the correspondence $(\mathcal{H}, \pi_B)\colon B \leftarrow \mathbb{C}$ and so $1 \otimes \pi_C(u) \in \mathcal{U}(\mathcal{E} \otimes_{\pi_B} \mathcal{H})$. This acts by $x \otimes \eta \mapsto x \otimes \pi_C(u)\eta$ for $x \in \mathcal{E}$, $\eta \in \mathcal{H}$. Since any element of $\mathcal{M}(C)$ is a linear combination of unitaries, there are well-defined bounded linear operators on $\mathcal{K}$ that act on elementary tensors by $x \otimes \eta \mapsto x \otimes \pi_C(c)\eta$ for $x \in \mathcal{E}$, $\eta \in \mathcal{H}$, $c \in C$. This defines a unital *-homomorphism $\pi'_C\colon \mathcal{M}(C) \to \mathbb{B}(\mathcal{K})$, which restricts to a nondegenerate *-homomorphism on C because $\pi_C(C)\mathcal{H} = \mathcal{H}$.

Realise $\mathcal{F}$ concretely as operators $\mathcal{K} \to \mathcal{L} := \mathcal{F} \otimes_{\pi'_C} \mathcal{K} = \mathcal{F} \otimes_{\pi'_C} (\mathcal{E} \otimes_{\pi_B} \mathcal{H})$ (compare Lemma 1.17). Now $x \in \mathcal{E}$ and $y \in \mathcal{F}$ give an operator $yx\colon \mathcal{H} \to \mathcal{K} \to \mathcal{L}$, mapping $\eta \mapsto y \otimes x \otimes \eta$ for each $\eta \in \mathcal{H}$. We compute

$$(y_1x_1)^*(y_2x_2) = x_1^*\pi'_C(y_1^*y_2)x_2 = \pi_C(y_1^*y_2)\pi_B(x_1^*x_2) = \pi(\langle x_1 \otimes y_1 \mid x_2 \otimes y_2\rangle)$$

for all $x_1, x_2 \in \mathcal{E}$, $y_1, y_2 \in \mathcal{F}$. Hence the map $x \otimes y \mapsto yx$ represents the inner product on $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ in a way that makes its positivity manifest.

Now assume the C*-norm on $B \otimes C$ to be a cross norm, that is, $\|b \otimes c\| = \|b\| \cdot \|c\|$ for all $b \in B$, $c \in C$. Then it dominates the spatial C*-tensor product norm by Takesaki's Theorem (see [6, Theorem 3.4.8]). Let π_B and π_C be faithful representations of B and C on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The tensor product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with the representation $\pi(b \otimes c) = \pi_B(b) \otimes \pi_C(c)$ is a faithful representation of the spatial tensor product by [6, Theorem 3.3.11]. Then π is also a representation of $B \otimes C$. Let $\mathcal{K}_1 := \mathcal{E} \otimes_B \mathcal{H}_1$ and $\mathcal{K}_2 := \mathcal{F} \otimes_C \mathcal{H}_2$. Represent $\mathcal{E}$ and $\mathcal{F}$ concretely in $\mathbb{B}(\mathcal{H}_j, \mathcal{K}_j)$ for $j = 1, 2$, respectively. Then $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ with its pre-Hilbert module structure over the spatial tensor product is represented concretely by operators $\mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{K}_1 \otimes \mathcal{K}_2$. This representation of the algebraic tensor product is easily seen to be faithful. This implies that the $B \otimes C$-valued inner product on $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ is nondegenerate. □

Example 5.3. If $\mathcal{E}$ is a Hilbert B-module and $\mathcal{H}$ is a Hilbert space, then there is a tensor product Hilbert B-module $\mathcal{E} \otimes \mathcal{H}$ because $B \otimes \mathbb{C} \cong B$. This gives the Hilbert modules in Example 1.10: $B \otimes \mathbb{C}^n = B^n$ and $B \otimes \ell^2\mathbb{N} \cong B^\infty$. The following lemma about such Hilbert modules is sometimes useful if $\mathcal{H} = L^2(X, \mu)$.

Lemma 5.4. *Let B be a* C*-*algebra and $\mathcal{E}$ a Hilbert B-module. Let X be a locally compact space and let μ be a Radon measure on X. Then the exterior product Hilbert B-module $L^2(X, \mu) \otimes \mathcal{E}$ is canonically isomorphic to the completion of* $\mathrm{C_c}(X, \mathcal{E})$*, the space of continuous, compactly supported functions $X \to \mathcal{E}$, with the right B-module structure by pointwise multiplication and the inner product*

$$\langle f_1 \mid f_2\rangle_B := \int_X f_1(x)^* f_2(x) \,\mathrm{d}\mu(x).$$

Proof. The subspace $\mathrm{C_c}(X)$ is dense in $L^2(X, \mu)$. Therefore, the algebraic tensor product $\mathrm{C_c}(X) \otimes_{\mathrm{alg}} \mathcal{E}$ is dense in $L^2(X, \mu) \otimes \mathcal{E}$. Define $\iota\colon \mathrm{C_c}(X) \otimes_{\mathrm{alg}} \mathcal{E} \to \mathrm{C_c}(X, \mathcal{E})$ by $\iota(f \otimes a)(x) := f(x) \cdot a$. The map ι has dense range by a partition of unity argument. More precisely, if $K \subseteq X$ is compact and $f \in \mathrm{C_c}(X, \mathcal{E})$ is supported in K, then there is a sequence $(f_n)_{n \in \mathbb{N}}$ with $f_n \in \mathrm{C_c}(X) \otimes_{\mathrm{alg}} \mathcal{E}$ and $\operatorname{supp} f_n \subseteq K$ for all $n \in \mathbb{N}$ and $\lim \|f - f_n\|_\infty = 0$. The inner product on $L^2(X, \mu) \otimes \mathcal{E}$, restricted to $\mathrm{C_c}(X) \otimes_{\mathrm{alg}} \mathcal{E}$, is given by the formula

$$\langle f_1 \otimes a_1 \mid f_2 \otimes a_2\rangle = \int_X \overline{f_1(x)} f_2(x) a_1^* a_2 \,\mathrm{d}\mu = \int_X \iota(f_1 \otimes a_1)^* \iota(f_2 \otimes a) \,\mathrm{d}\mu.$$

Since all expressions here are sesquilinear, this extends from elementary tensors to all elements of $\mathrm{C_c}(X) \otimes_{\mathrm{alg}} \mathcal{E}$. That is, ι restricted to $\mathrm{C_c}(X) \otimes_{\mathrm{alg}} \mathcal{E}$ is an isometric linear map for the inner products defined on $\mathrm{C_c}(X, \mathcal{E})$ and $L^2(X, \mu) \otimes \mathcal{E}$.

If $f \in \mathrm{C_c}(X,\mathcal{E})$ and $(f_n)_{n\in\mathbb{N}}$ are as above, then a computation shows that the sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence for the norm on the Hilbert module $L^2(X,\mu)\otimes\mathcal{E}$. Mapping f to the limit of this Cauchy sequence defines a map $\iota\colon \mathrm{C_c}(X,\mathcal{E}) \to L^2(X,\mu)\otimes\mathcal{E}$ that is still isometric. Since $\iota(\mathrm{C_c}(X)\otimes_{\mathrm{alg}}\mathcal{E})$ is dense in $L^2(X,\mu)\otimes\mathcal{E}$, it follows that $L^2(X,\mu)\otimes\mathcal{E}$ is the completion of $\mathrm{C_c}(X,\mathcal{E})$ in the inner product defined in the statement of the lemma. □

Exercise 5.5. *In the situation of Lemma 5.4, let $L^2(X,A)_2$ be the space of all measurable functions $f\colon X\to A$ for which $\int_X \|f(x)\|^2\,\mathrm{d}\mu$ is finite. Show that the isometric embedding $\mathrm{C_c}(X,A)\to L^2(X)\otimes A$ extends to an embedding of $L^2(X,A)_2$ into $L^2(X)\otimes A$ with dense range.*

Remark 5.6. It is not true that elements of the Hilbert module $L^2(X,\mu)\otimes A$ may be represented by measurable functions $X\to A$. Only a dense set of elements in $L^2(X,\mu)\otimes A$ has this property. There may, however, be nets in $L^2(X,\mu)\otimes_{\mathrm{alg}} A$ or $\mathrm{C_c}(X,A)$ that converge in the Hilbert module norm of $L^2(X,\mu)\otimes A$ and whose values at $x\in X$ fail to converge for almost all $x\in X$. This makes it a bit tricky to define adjointable operators on $L^2(X,\mu)\otimes A$. The best way is usually to define them first as operators on $\mathrm{C_c}(X,A)$ or $L^2(X,A)_2$ or a similar dense subspace of functions $X\to A$, then to check boundedness for the Hilbert module norm, and then to extend to $L^2(X,\mu)\otimes A$ (compare Lemma 5.9).

Proposition 5.7. *Let $\mathcal{E}$ and $\mathcal{F}$ be correspondences $A \xleftarrow{\mathcal{C}} B$ and $D \xleftarrow{\mathcal{C}} C$. Then $\mathcal{E}\otimes_{\mathrm{alg}}\mathcal{F}$ completes to a C^*-correspondence between the maximal tensor products $A\otimes_{\max} D$ and $B\otimes_{\max} C$ and to a C^*-correspondence between the spatial tensor products $A\otimes_{\min} D$ and $B\otimes_{\min} C$.*

Proof. Equip $\mathcal{E}\otimes_{\mathrm{alg}}\mathcal{F}$ with the obvious left actions of A and D. The same argument as in the proof of Lemma 2.1 shows that these left actions extend to (nondegenerate) representations of A and D on the Hilbert $B\otimes C$-module completion $\mathcal{E}\otimes\mathcal{F}$. These representations have commuting ranges and hence give a representation of the maximal tensor product $A\otimes_{\max} D$. Thus the completion of $\mathcal{E}\otimes_{\mathrm{alg}}\mathcal{F}$ becomes a concrete correspondence $A\otimes_{\max} D \xleftarrow{\mathcal{C}} B\otimes_{\max} C$.

For the spatial C^*-tensor product $B\otimes_{\min} C$, we may realise $\mathcal{E}\otimes\mathcal{F}$ concretely as in the last paragraph of the proof of Lemma 5.2. The induced representations of A on $\mathcal{K}_1$ and of B on $\mathcal{K}_2$ combine to a representation of $A\otimes_{\min} D$ on $\mathcal{K}_1\otimes\mathcal{K}_2$. Thus $\mathcal{E}\otimes_{\mathrm{alg}}\mathcal{F}$ becomes a concrete correspondence $A\otimes_{\min} D \xleftarrow{\mathcal{C}} B\otimes_{\min} C$. □

For any C^*-completion $B\otimes C$, embeddings $\mathcal{E}_1\hookrightarrow\mathcal{E}_2$ and $\mathcal{F}_1\hookrightarrow\mathcal{F}_2$ induce an embedding $\mathcal{E}_1\otimes\mathcal{E}_2\hookrightarrow\mathcal{F}_1\otimes\mathcal{F}_2$. Thus the exterior product gives a functor

$$\otimes\colon \mathfrak{Corr}(A,B)\times\mathfrak{Corr}(D,C)\to\mathfrak{Corr}(A\otimes D, B\otimes C)$$

for all C^*-algebras A,B,C,D, any C^*-tensor product $B\otimes C$, and a suitable tensor product $A\otimes D$ depending on $B\otimes C$. In particular, $\otimes$ may be $\otimes_{\max}$ or $\otimes_{\min}$ both times.

Theorem 5.8. *Both the maximal and spatial tensor products define strictly unital bicategory homomorphisms $\mathfrak{Corr}\times\mathfrak{Corr}\to\mathfrak{Corr}$.*

Proof. We have already seen that both exterior products give functors

$$\mathfrak{Corr}(A,B)\times\mathfrak{Corr}(D,C)\to\mathfrak{Corr}(A\otimes_{\max}D, B\otimes_{\max}C),\qquad \mathfrak{Corr}(A\otimes_{\min}D, B\otimes_{\min}C).$$

Let $\mathcal{E}_1\colon B_1\xleftarrow{\mathcal{C}} B_2$, $\mathcal{E}_2\colon B_2\xleftarrow{\mathcal{C}} B_3$, $\mathcal{F}_1\colon C_1\xleftarrow{\mathcal{C}} C_2$ and $\mathcal{F}_2\colon C_2\xleftarrow{\mathcal{C}} C_3$. Then there is a canonical isomorphism

$$(\mathcal{E}_1\otimes\mathcal{F}_1)\otimes_{B_2\otimes C_2}(\mathcal{E}_2\otimes\mathcal{F}_2)\cong(\mathcal{E}_1\otimes_{B_2}\mathcal{E}_2)\otimes(\mathcal{F}_1\otimes_{C_2}\mathcal{F}_2);$$

here the C*-tensor norms on $B_i \otimes C_i$ may be $\otimes_{\max}$ everywhere or $\otimes_{\min}$ everywhere, or they may be another suitably compatible choice. The exterior product of two identity correspondences is again an identity correspondence. We have described the data for a homomorphism of bicategories. It is easy to check the coherence conditions for a homomorphism. $\square$

Both the maximal and the spatial tensor products turn C*-algebras into a symmetric monoidal category. That is, there are natural isomorphisms

$$(A\otimes_{\max} B)\otimes_{\max} C \cong A\otimes_{\max}(B\otimes_{\max} C), \quad A\otimes_{\max} B \cong B\otimes_{\max} A, \quad A\otimes_{\max}\mathbb{C} \cong A,$$

which satisfy the coherence conditions for a symmetric monoidal category, and similarly for $\otimes_{\min}$. It follows from Proposition 5.1 that the direct sum of C*-algebras embeds in a bicategory homomorphism on $\mathfrak{Corr}$. The tensor product is infinitely distributive over the maximal and spatial tensor products, that is, there are natural isomorphisms of C*-algebras

$$\bigoplus_{i\in I} (A \otimes_{\max} B_i) \cong A \otimes_{\max} \bigoplus_{i\in I} B_i, \qquad \bigoplus_{i\in I} (A \otimes_{\min} B_i) \cong A \otimes_{\min} \bigoplus_{i\in I} B_i.$$

All these natural C*-algebra isomorphisms seem nicely compatible with the bicategory structure on $\mathfrak{Corr}$. I guess that the direct sum and the spatial and the maximal tensor product each make the correspondence bicategory into a monoidal bicategory, which is a tricategory with one object (see [10]). I have not yet needed this higher structure and therefore have not tried to check the coherence conditions for a tricategory in [10].

As a first application of the exterior product, we prove that the reduced crossed product for a group action does not depend on the choice of a faithful representation of A. A similar argument shows that the spatial C*-tensor product does not depend on the faithful representations of the tensor factors. In fact, we shall define the reduced crossed product for twisted actions of locally compact groups in a representation-free way using a Hilbert module over the coefficient algebra instead of Hilbert space representations.

Lemma 5.9. *Let G be a locally compact group, let A be a C*-algebra, and let $\alpha\colon G \to \operatorname{Aut}(A)$ and $u\colon G \times G \to \mathcal{U}(A)$ form a Borel twisted action of G on A. There are a nondegenerate *-homomorphism $\varrho\colon A \to \mathbb{B}(L^2(G) \otimes A)$ and a Borel map $V\colon G \to \mathcal{U}(L^2(G)\otimes A)$ defined by $\varrho(a)(f)(x) := \alpha_{x^{-1}}(a)\cdot f(x)$ and $(V_g f)(x) := u^*_{x^{-1},g} f(g^{-1}x))$ for all $f \in L^2(G, A)_2$, $g, x \in G$, $a \in A$. The pair (ϱ, V) is a covariant representation of the twisted action of G on A.*

Proof. It is clear that $\varrho(a)$ for $a \in \mathcal{M}(A)$ defines a bounded linear map on $L^2(G, A)_2$ and that V_g defines an isometric linear map on $L^2(G, A)_2$. The space $L^2(G, A)_2$ is isomorphic to a dense subspace of $L^2(G) \otimes A$ by Exercise 5.5. Easy computations shows that V_g for $g \in G$ and $\varrho(u)$ for $u \in \mathcal{U}(A)$ are unitary operators for the inner product of $L^2(G) \otimes A$. Therefore, these operators defined initially on $L^2(G, A)_2$ extend uniquely to unitary operators on $L^2(G) \otimes A$. Since unitaries span $\mathcal{M}(A)$, it follows that $\varrho(a)$ extends to an adjointable operator on $L^2(G) \otimes A$ for all $a \in A$. The map $\varrho\colon A \to \mathbb{B}(L^2(G) \otimes A)$ is easily seen to be a *-homomorphism. We check the covariance condition:

$$\begin{aligned}(\varrho(\alpha_g(a))V_g f)(x) &= \alpha_{x^{-1}}(\alpha_g(a)) \cdot u^*_{x^{-1},g} \cdot f(g^{-1}x)\\ &= u^*_{x^{-1},g} \cdot \alpha_{x^{-1}g}(a) \cdot f(g^{-1}x) = (V_g \varrho(a) f)(x)\end{aligned}$$

for all $g, x \in G$, $a \in A$, $f \in L^2(G, A)_2$; this implies $\varrho(\alpha_g(a))V_g = V_g\varrho(a)$ in $\mathbb{B}(L^2(G) \otimes A)$ for all $g \in G$. We integrate the covariant pair (ϱ, V) to a *-homomorphism $\psi\colon L^1(G, A) \to \mathbb{B}(L^2(G) \otimes A)$. Here

$$(\psi(f)h)(x) = \int_G \alpha_{x^{-1}}(f(g)) \cdot u^*_{x^{-1},g} \cdot h(g^{-1}x)\,\mathrm{d}g$$

for $f \in L^1(G, A)$ $h \in L^2(G, A)_2$, $x \in G$. If α is a continuous group action, then it is easy to prove that the representation ϱ is nondegenerate. If α is only Borel, then we omit this argument. Using the approximate unit in the crossed product Banach algebra $L^1(G, A)$, it is not hard to see that ψ is nondegenerate. And this implies that ϱ is nondegenerate. □

Definition 5.10. The covariant representation of (A, G, α, u) on $L^2(G) \otimes A$ in Lemma 5.9 is called the *regular representation.* It integrates to a nondegenerate representation $A \rtimes_{\alpha,u} G \to \mathbb{B}(L^2(G) \otimes A)$. We define the *reduced crossed product* $A \rtimes_{\mathrm{r},\alpha,u} G$ to be the image of this representation. Equivalently, it is the norm closure of the image of the Banach algebra crossed product $L^1(G, A)$ in the norm of $\mathbb{B}(L^2(G) \otimes A)$.

Remark 5.11. The representation $L^1(G, A) \to \mathbb{B}(L^2(G) \otimes A)$ obtained from the regular covariant representation is always faithful. Therefore, $L^1(G, A)$ maps faithfully into the reduced crossed product.

Now let $\pi\colon A \to \mathbb{B}(\mathcal{H})$ be a faithful representation of A on a Hilbert space. Let

$$\mathcal{K} := (L^2(G) \otimes A) \otimes_A \mathcal{H}.$$

The map $(L^2(G)\otimes_{\mathrm{alg}} A)\otimes_{\mathrm{alg}}\mathcal{H} \to L^2(G)\otimes_{\mathrm{alg}}\mathcal{H}$, $(f\otimes a)\otimes\xi \mapsto f\otimes\pi(a)\xi$, is isometric for the inner product on $(L^2(G)\otimes A)\otimes_A\mathcal{H}$ and the usual Hilbert space tensor product $L^2(G) \otimes \mathcal{H}$. It also has dense range because all representations are nondegenerate by convention. This identifies $\mathcal{K} \cong L^2(G) \otimes \mathcal{H} = L^2(G, \mathcal{H})$. Theorem 3.16 identifies $\mathbb{B}(L^2(G) \otimes A)$ isometrically with a C*-subalgebra of $\mathbb{B}(L^2(G, \mathcal{H}))$. The covariant representation (ϱ, V) on $L^2(G) \otimes A$ induces a covariant representation (ϱ', V') on the Hilbert space $L^2(G, \mathcal{H})$, which is given by essentially the same formulas as (ϱ, V). Namely, $(\varrho'(a)f)(x) = \pi(\alpha_{x^{-1}}(a)) \cdot f(x)$ and $(V_g f)(x) := u^*_{x^{-1},g} f(g^{-1}x)$ for all $f \in L^2(G, \mathcal{H})$, $g, x \in G$, $a \in A$. The covariant representation (ϱ', V') integrates to a representation $A \rtimes_{\alpha,u} G \to \mathbb{B}(L^2(G, \mathcal{H}))$. The image of this representation is contained in $\mathbb{B}(L^2(G) \otimes A)$. So the operator norm on $L^2(G, \mathcal{H})$ and on the Hilbert module $L^2(G) \otimes A$ give the same C*-seminorm on $A \rtimes_{\alpha,u} G$. Therefore, we may equivalently define $A \rtimes_{\alpha,u} G$ using the covariant representation (ϱ', V') on $L^2(G, \mathcal{H})$. This is the usual definition of the reduced crossed product. It has the advantage of using only Hilbert space representations. It has the disadvantage that it needs a faithful representation of A as input data. It is nontrivial to prove that the reduced norm on $L^1(G, A)$ does not depend on this choice.

As another application of the exterior product, we prove that the inner product on $\mathcal{E} \otimes_B \mathcal{H}$ is nondegenerate if $\mathcal{E}$ is a Hilbert module and $\mathcal{H}$ a Hilbert space; this was claimed in Remark 1.14.

Example 5.12. We may view $\mathbb{C}^n$ as a correspondence $\mathbb{M}_n(\mathbb{C}) \xleftarrow{\mathcal{C}} \mathbb{C}$. Therefore, if $\mathcal{F}$ is a correspondence $B \xleftarrow{\mathcal{C}} C$, then $\mathcal{F}^n = \mathbb{C}^n \otimes \mathcal{F}$ is a correspondence $\mathbb{M}_n(B) = \mathbb{M}_n \otimes B \xleftarrow{\mathcal{C}} \mathbb{C} \otimes C = C$ (see Example 1.11). Since $\mathbb{C}^n\colon \mathbb{M}_n(\mathbb{C}) \xleftarrow{\mathcal{C}} \mathbb{C}$ is an equivalence, there is the inverse equivalence $(\mathbb{C}^n)^*\colon \mathbb{C} \xleftarrow{\mathcal{C}} \mathbb{M}_n(\mathbb{C})$. Therefore, if $\mathcal{E}$ is a correspondence $A \xleftarrow{\mathcal{C}} B$, then $\mathcal{E}^n \cong (\mathbb{C}^n)^* \otimes \mathcal{E}$ is a correspondence $A \xleftarrow{\mathcal{C}} \mathbb{M}_n(B)$. The inner product of two vectors (x_i), (y_i) is the matrix with entries $\langle x_i \,|\, y_j\rangle_B$. The positivity of this inner product for all $n \in \mathbb{N}$ is exactly Lemma 1.4. Since the exterior tensor product is compatible with the composition of correspondences, $\mathcal{E}^n \otimes_{\mathbb{M}_n(B)} \mathcal{F}^n \cong \mathcal{E} \otimes_B \mathcal{F}$.

Proposition 5.13 (see [15, Proposition 4.5])**.** *Let $\mathcal{E}\colon A \leftarrow B$ and $\mathcal{F}\colon B \leftarrow C$ be correspondences. The inner product on $\mathcal{E} \odot_B \mathcal{F}$ is nondegenerate.*

Proof. Write $\xi \in \mathcal{E} \otimes \mathcal{F}$ as $\xi = \sum_{i=1}^n x_1 \otimes y_1$ with $\vec{x} := (x_1, \dots, x_n) \in \mathcal{E}^n$ and $\vec{y} := (y_1, \dots, y_n) \in \mathcal{F}^n$. View $\mathcal{E}^n$ and $\mathcal{F}^n$ as correspondences $\mathcal{E}^n\colon A \leftarrow \mathbb{M}_n(B)$ and $\mathcal{F}^n\colon \mathbb{M}_n(B) \leftarrow C$. Then

$$\begin{aligned}\langle \xi \mid \xi \rangle = \sum_{i,j=1}^n \langle y_i \mid \langle x_i \mid x_j \rangle_B y_j \rangle_C &= \langle \vec{y} \mid \langle \vec{x} \mid \vec{x} \rangle_{\mathbb{M}_n(B)} \vec{y} \rangle_C \\ &= \langle \langle \vec{x} \mid \vec{x} \rangle^{1/2}_{\mathbb{M}_n(B)} \vec{y} \mid \langle \vec{x} \mid \vec{x} \rangle^{1/2}_{\mathbb{M}_n(B)} \vec{y} \rangle_C.\end{aligned}$$

Thus $\langle \xi \mid \xi \rangle = 0$ implies $\langle \vec{x} \mid \vec{x} \rangle^{1/2}_{\mathbb{M}_n(B)} \vec{y} = 0$. Proposition 3.12 gives $z \in \mathcal{E}^n$ with $\vec{x} = z \langle z \mid z \rangle_{\mathbb{M}_n(B)}$. That is, $z = (z_1, \dots, z_n)$ and $x_i = \sum_{j=1}^n z_j \langle z_j \mid z_i \rangle_B$ for $i = 1, \dots, n$. Then $\langle \vec{x} \mid \vec{x} \rangle_{\mathbb{M}_n(B)} = \langle z \mid z \rangle^3_{\mathbb{M}_n(B)}$ and hence $\langle z \mid z \rangle_{\mathbb{M}_n(B)} \vec{y} = 0$ because

$$\langle \langle z \mid z \rangle \vec{y} \mid \langle z \mid z \rangle \vec{y} \rangle = \langle \langle \vec{x} \mid \vec{x} \rangle^{1/3} \vec{y} \mid \langle \vec{x} \mid \vec{x} \rangle^{1/3} \vec{y} \rangle = \langle \langle \vec{x} \mid \vec{x} \rangle^{1/6} \vec{y} \mid \langle \vec{x} \mid \vec{x} \rangle^{1/2} \vec{y} \rangle = 0.$$

The equation $\langle z \mid z \rangle_{\mathbb{M}_n(B)} \vec{y} = 0$ says $\sum_{i=1}^n \langle z_j \mid z_i \rangle_B y_i = 0$ for $j = 1, \dots, n$. Thus

$$\sum_{i=1}^n x_i \otimes y_i = \sum_{i,j=1}^n z_j \langle z_j \mid z_i \rangle_B \otimes y_i = \sum_{i,j=1}^n z_j \langle z_j \mid z_i \rangle_B \otimes y_i - z_j \otimes \langle z_j \mid z_i \rangle_B y_i.$$

Since $\langle z_j \mid z_i \rangle_B \in B$, this shows that every element in the null-space of the inner product is a linear combination of terms of the form $xb \otimes y - x \otimes by$ with $x \in \mathcal{E}$, $b \in B$, $\eta \in \mathcal{F}$. □

6. Morita–Rieffel equivalence

We are going to describe the equivalences in the correspondence bicategory. These are, by definition, the Morita–Rieffel equivalences. As a first step, we describe a candidate for the inverse of an equivalence.

Definition 6.1. Let $\mathcal{E}$ be a Hilbert B-module. Let $\mathcal{E}^* := \mathbb{K}(\mathcal{E}, B)$ with the right $\mathbb{K}(\mathcal{E})$-module structure by composition and with the inner product $\langle T \mid S \rangle := T^* \circ S \in \mathbb{K}(\mathcal{E})$. This is a Hilbert $\mathbb{K}(\mathcal{E})$-module. (The norm is the operator norm from $\mathbb{B}(\mathcal{E})$, so that $\mathcal{E}^*$ is complete.)

Lemma 6.2. *The operators $|x\rangle$ and $\langle x|$ are compact for all $x \in \mathcal{E}$. The maps*

$$\begin{aligned}\mathcal{E} &\xrightarrow{\simeq} \mathbb{K}(B, \mathcal{E}), \qquad x \mapsto |x\rangle, \\ \overline{\mathcal{E}} &\xrightarrow{\simeq} \mathbb{K}(\mathcal{E}, B), \qquad x \mapsto \langle x|,\end{aligned}$$

are isometric isomorphisms of Banach spaces, where $\overline{\mathcal{E}}$ denotes the conjugate Banach space of $\mathcal{E}$.

Proof. The space $\mathbb{K}(B, \mathcal{E})$ is the closed linear span of the operators $|x\rangle\langle a|$, $b \mapsto x \cdot a^* \cdot b$, for all $x \in \mathcal{E}$, $a, b \in B$. Since $|x\rangle\langle a| = |xa^*\rangle$ and $\mathcal{E} \cdot B = \mathcal{E}$ by Corollary 3.13, $|x\rangle$ is compact for $x \in \mathcal{E}$ and the map $x \mapsto |x\rangle$ is an isomorphism $\mathcal{E} \xrightarrow{\simeq} \mathbb{K}(B, \mathcal{E})$. This is isometric by the C*-property of the norm. Then the adjoint operators $\langle x| = |x\rangle^*$ are compact as well and give an isomorphism $\overline{\mathcal{E}} \cong \mathbb{K}(\mathcal{E}, B)$. □

Thus we could also have defined $\mathcal{E}^*$ as the conjugate Banach space with the $\mathbb{K}(\mathcal{E})$-module structure $\overline{x} \cdot T = \overline{T^*(x)}$ and the inner product $\langle \overline{x} \mid \overline{y} \rangle_{\mathbb{K}(\mathcal{E})} = |y\rangle\langle x|$ for $x, y \in \mathcal{E}$, $T \in \mathbb{K}(\mathcal{E})$.

Exercise 6.3. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert B-modules. View $\mathcal{E}_1^*$ as a correspondence $B \leftarrow_{\mathcal{C}} \mathbb{K}(\mathcal{E}_1)$; view $\mathcal{E}_2$ as a correspondence $\mathbb{K}(\mathcal{E}_2) \leftarrow_{\mathcal{C}} B$, and view $\mathbb{K}(\mathcal{E}_1, \mathcal{E}_2)$ as a correspondence $\mathbb{K}(\mathcal{E}_2) \leftarrow_{\mathcal{C}} \mathbb{K}(\mathcal{E}_1)$ (see also Exercise 7.17). Prove*

$$\mathbb{K}(\mathcal{E}_1, \mathcal{E}_2) \cong \mathcal{E}_2 \otimes_B \mathcal{E}_1^*.$$

Theorem 6.4. *A correspondence $\mathcal{E}\colon A \leftarrow_{\mathcal{C}} B$ is an equivalence if and only if*

- *$\mathcal{E}$ is a full Hilbert B-module;*
- *the left action $\alpha\colon A \to \mathbb{B}(\mathcal{E})$ is an isomorphism onto $\mathbb{K}(\mathcal{E})$.*

In this case, the inverse of $\mathcal{E}$ is $\mathcal{E}^*$ with the inner product $\langle x \mid y\rangle_A := \alpha^{-1}(|x\rangle\langle y|)$ for $x, y \in \mathcal{E}$ and the B,A-module structure $b \cdot T \cdot a := bT\alpha(a)$, and the correspondence isomorphisms $\mathcal{E} \otimes_B \mathcal{E}^* \cong A$ and $\mathcal{E}^* \otimes_A \mathcal{E} \cong B$ are induced by the inner products on $\mathcal{E}^*$ and $\mathcal{E}$, respectively.

This theorem is equivalent to [8, Proposition 2.6].

Proof. First assume that $\mathcal{E}$ is a full Hilbert module and that the left action is an isomorphism $A \xrightarrow{\cong} \mathbb{K}(\mathcal{E})$. Then $\mathcal{E}^*$ becomes a correspondence $\mathcal{E}^*\colon B \leftarrow_{\mathcal{C}} A$ as in the statement of the theorem. We claim that there is an isomorphism of correspondences

$$\mu\colon \mathcal{E} \otimes_B \mathcal{E}^* \to A, \qquad x \otimes \xi \mapsto \alpha^{-1}(|x\rangle \circ \xi).$$

This formula well-defines an isometric map because

$$\begin{aligned}\big\langle |x_1\rangle \circ \xi_1 \,\big|\, |x_2\rangle \circ \xi_2 \big\rangle_A &= \alpha^{-1}(\xi_1^* \langle x_1 \mid x_2\rangle_B \xi_2) \\ &= \big\langle \xi_1 \,\big|\, \langle x_1 \mid x_2\rangle_B \xi_2 \big\rangle_A = \langle x_1 \otimes \xi_1 \mid x_2 \otimes \xi_2\rangle_A.\end{aligned}$$

Since $\langle y| \in \mathcal{E}^*$ for all $y \in \mathcal{E}$ and α is an isomorphism, $\mu(\mathcal{E} \otimes_B \mathcal{E}^*) \ni \alpha^{-1}(|x\rangle\langle y|)$ for $x, y \in \mathcal{E}$, which is dense in A. Hence μ is an isomorphism of correspondences $\mathcal{E} \otimes_B \mathcal{E}^* \cong A$. Similarly, we define a map

$$\mu'\colon \mathcal{E}^* \otimes_A \mathcal{E} \to B, \qquad \xi \otimes x \mapsto \xi(x).$$

It is isometric because

$$\langle \xi \otimes x \mid \xi \otimes x\rangle_B = \langle x \mid (\xi^*\xi)(x)\rangle_B = \langle \xi(x) \mid \xi(x)\rangle_B.$$

The range of μ' contains the closed linear span of $\langle y|(x) = \langle y \mid x\rangle_B$ for all $x, y \in \mathcal{E}$. Since $\mathcal{E}$ is full, μ' is surjective. Hence $\mathcal{E}^* \otimes_A \mathcal{E} \cong B$. Thus $\mathcal{E}$ is an equivalence in the correspondence bicategory if $\mathcal{E}$ is full and α is an isomorphism onto $\mathbb{K}(\mathcal{E})$.

Conversely, assume that $\mathcal{E}$ is an equivalence in the correspondence bicategory. Let $\mathcal{F}$ be a correspondence $B \leftarrow_{\mathcal{C}} A$ with $\mathcal{E} \otimes_B \mathcal{F} \cong A$ and $\mathcal{F} \otimes_A \mathcal{E} \cong B$. Then $\mathcal{E}$ is full because

$$B = \langle B \mid B\rangle_B = \langle \mathcal{F} \otimes_A \mathcal{E} \mid \mathcal{F} \otimes_A \mathcal{E}\rangle_B = \langle \mathcal{E} \mid \langle \mathcal{F} \mid \mathcal{F}\rangle_A \mathcal{E}\rangle_B \subseteq \langle \mathcal{E} \mid \mathcal{E}\rangle_B.$$

When we view $\mathcal{E}$ as a correspondence $\mathbb{K}(\mathcal{E}) \leftarrow_{\mathcal{C}} B$, it becomes an equivalence in $\mathfrak{Corr}$. We compose $\mathcal{E}\colon A \leftarrow_{\mathcal{C}} B$ with the equivalence $\mathcal{E}^*\colon B \leftarrow_{\mathcal{C}} B' := \mathbb{K}(\mathcal{E})$. This gives an equivalence $\mathcal{E} \otimes_B \mathcal{E}^*\colon A \leftarrow_{\mathcal{C}} B'$ in $\mathfrak{Corr}$ as a product of two equivalences. The multiplication map μ above identifies $\mathcal{E} \otimes_B \mathcal{E}^* \cong \mathbb{K}(\mathcal{E})$, and the left action of A becomes left multiplication with $\alpha(a)$. Thus the equivalence $A \leftarrow_{\mathcal{C}} B'$ is the correspondence $\mathbb{K}(\mathcal{E})_\alpha$ associated to the morphism $\alpha\colon A \to \mathbb{K}(\mathcal{E})$.

The inverse correspondence $\mathcal{F}'\colon B' \leftarrow_{\mathcal{C}} A$ must satisfy $\mathcal{F}' \otimes_A B' \cong B'$ and $B' \otimes_{B'} \mathcal{F}' \cong A$ as correspondences. The canonical isomorphism $B' \otimes_{B'} \mathcal{F}' \cong \mathcal{F}'$ in Lemma 4.4 shows that $\mathcal{F}' \cong A$ as a Hilbert A-module, with B' acting by some morphism $\beta\colon B' \to A$. The correspondences associated to the composite morphisms $\beta \circ \alpha$ and $\alpha \circ \beta$ are isomorphic to the identity correspondences by assumption. These isomorphisms say that there are $u \in \mathcal{U}(A)$, $v \in \mathcal{U}(B)$ with $\beta \circ \alpha = \mathrm{Ad}_u$, $\alpha \circ \beta = \mathrm{Ad}_v$. Then α is both right and left invertible in the morphism category. Then α is a *-isomorphism $A \cong B' = \mathbb{K}(\mathcal{E})$. $\square$

Definition 6.5. Two C*-algebras A and B are *Morita–Rieffel equivalent* if there is a full Hilbert B-module $\mathcal{E}$ with $\mathbb{K}(\mathcal{E}) \cong A$.

Corollary 6.6. *Morita–Rieffel equivalence for* C*-*algebras is an equivalence relation.*

Proof. This follows from Theorem 6.4. □

The proof of Theorem 6.4 used the following example:

Example 6.7. A full correspondence $\mathcal{E}\colon A \leftarrow B$ gives an equivalence from $\mathbb{K}(\mathcal{E})$ to B. The left action of A on $\mathbb{K}(\mathcal{E})$ gives a morphism $A \to \mathbb{K}(\mathcal{E})$. The correspondence $\mathcal{E}$ is the composite of the morphism $A \to \mathbb{K}(\mathcal{E})$ and the equivalence $\mathbb{K}(\mathcal{E}) \simeq B$. Similarly, a full and proper correspondence $\mathcal{E}$ decomposes into a proper morphism $A \to \mathbb{K}(\mathcal{E})$ and a Morita–Rieffel equivalence.

Theorem 6.4 implies that all equivalences in $\mathfrak{Corr}$ are full, proper and faithful and hence equivalences in the subbicategory of full, proper and faithful correspondences.

We are going to formulate Morita–Rieffel equivalence more symmetrically in Section 7. This alternative formulation makes it easy to prove that Morita–Rieffel equivalent C*-algebras have isomorphic ideal lattices. Theorem 6.4 shows that Morita–Rieffel equivalences are proper correspondences.

Remark 6.8. Proper correspondences induce maps on K-theory, and isomorphic proper correspondences induce the same map on K-theory. Since a Morita–Rieffel equivalence is an equivalence in the bicategory of proper correspondences, it induces an isomorphism on K-theory. For σ-unital C*-algebras, this follows also from the Brown–Green–Rieffel Theorem below (Theorem 10.8).

7. Hilbert bimodules

Let $\mathcal{E}\colon A \leftarrow B$ be a Morita–Rieffel equivalence. The left action on $\mathcal{E}$ is through an isomorphism $\alpha\colon A \xrightarrow{\cong} \mathbb{K}(\mathcal{E})$. Thus there is an A-valued inner product

$$\langle\langle x \mid y\rangle\rangle_A := \alpha^{-1}\big(|x\rangle\langle y|\big)$$

for $x, y \in \mathcal{E}$. This inner product and the left A-module structure α turn $\mathcal{E}$ into a *left* Hilbert A-module. That is, the A-valued inner product is linear in the *first* variable, $\langle\langle a \cdot x_1 \mid x_2\rangle\rangle_A = a \cdot \langle\langle x_1 \mid x_2\rangle\rangle_A$ for all $a \in A$, $x_1, x_2 \in \mathcal{E}$, and it satisfies analogues of (3)–(5) in Definition 1.2. In addition, the left and right Hilbert module structures are compatible in the following sense:

(1) $\mathcal{E}$ is an A, B-bimodule;
(2) $\langle\langle x \mid y\rangle\rangle_A z = x\langle y \mid z\rangle_B$ for all $x, y, z \in \mathcal{E}$.

Definition 7.1. Let A and B be C*-algebras. A *Hilbert A, B-bimodule* is a set $\mathcal{E}$ with a left Hilbert A-module and a right Hilbert B-module structure, subject to the two compatibility conditions above.

Proposition 7.2. *A correspondence $\mathcal{E}$ carries a Hilbert bimodule structure if and only if there is an ideal $I \triangleleft A$ such that the left action $\alpha\colon A \to \mathbb{B}(\mathcal{E})$ restricts to an isomorphism from I onto $\mathbb{K}(\mathcal{E})$. The correspondence pins down this ideal and the left inner product, and $I = \langle\langle \mathcal{E} \mid \mathcal{E}\rangle\rangle_A$. A correspondence embedding between two Hilbert bimodules is automatically a Hilbert bimodule map.*

Proof. Let $\mathcal{E}$ be a Hilbert bimodule. Let $I := \langle\langle \mathcal{E} \mid \mathcal{E}\rangle\rangle \triangleleft A$. Since

$$\langle\langle x_1 \mid x_2\rangle\rangle_A x_3 = x_1\langle x_2 \mid x_3\rangle_B = |x_1\rangle\langle x_2| x_3$$

for all $x_1, x_2, x_3 \in \mathcal{E}$ and the operators of the form $|x_1\rangle\langle x_2|$ span $\mathbb{K}(\mathcal{E})$, the left action maps the ideal I onto $\mathbb{K}(\mathcal{E})$. If $a \in A$ satisfies $\alpha(a) = 0$, then $a\langle\langle x_1 \mid x_2\rangle\rangle_A =$

$\langle\!\langle ax_1 \mid x_2\rangle\!\rangle_A = 0$ for all $x_1, x_2 \in \mathcal{E}$, so that $a \cdot I = 0$. Thus $\alpha|_I$ is injective, and then it maps I isomorphically onto $\mathbb{K}(\mathcal{E})$.

Conversely, if $I \triangleleft A$ and $\alpha|_I \colon I \xrightarrow{\simeq} \mathbb{K}(\mathcal{E})$ is an isomorphism, then $\langle\!\langle x_1 \mid x_2\rangle\!\rangle := \alpha|_I^{-1}(|x_1\rangle\langle x_2|)$ for $x_1, x_2 \in \mathcal{E}$ defines a left I-valued inner product that turns $\mathcal{E}$ into a Hilbert A, B-bimodule because $I \triangleleft A$.

If $I' \triangleleft A$ is another ideal with $\alpha(I') = \mathbb{K}(\mathcal{E})$, then $\alpha(I \cdot I') = \mathbb{K}(\mathcal{E}) \cdot \mathbb{K}(\mathcal{E}) = \mathbb{K}(\mathcal{E})$ as well. Since $\alpha|_I$ is an isomorphism onto $\mathbb{K}(\mathcal{E})$, $I \cdot I'$ cannot be smaller than I. Thus I is the minimal ideal that α maps onto $\mathbb{K}(\mathcal{E})$, and the only one on which this happens isomorphically. Thus the underlying correspondence determines I.

Let $\mathcal{E}'$ be another Hilbert A, B-bimodule with the same underlying correspondence as $\mathcal{E}$ and with left A-valued inner product $\langle\!\langle x \mid y\rangle\!\rangle'_A$. Then

$$\alpha(\langle\!\langle x \mid y\rangle\!\rangle'_A)z = x \cdot \langle y \mid z\rangle_B = \alpha(\langle\!\langle x \mid y\rangle\!\rangle_A)z$$

for all $x, y, z \in \mathcal{E}$. Since $I = \langle\!\langle \mathcal{E} \mid \mathcal{E}\rangle\!\rangle'_A$ as well and the restriction of α to I is faithful, our two left inner products are equal. So the left inner product is unique.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Hilbert bimodules and let $i \colon \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ be an embedding of correspondences. Then $\langle\!\langle i\xi \mid i\eta\rangle\!\rangle_A$ and $\langle\!\langle \xi \mid \eta\rangle\!\rangle_A$ are two left inner products on $\mathcal{E}_1$ that make it a Hilbert bimodule. Hence they are equal. So i is automatically a Hilbert bimodule map. □

If $\alpha|_I \colon I \xrightarrow{\simeq} \mathbb{K}(\mathcal{E})$, then α is the morphism $A \to \mathbb{B}(\mathcal{E}) \cong \mathcal{M}(\mathbb{K}(\mathcal{E}))$ associated to the ideal inclusion $\mathbb{K}(\mathcal{E}) \cong I \triangleleft A$.

Theorem 7.3. *A Hilbert A, B-bimodule is the same as a Morita–Rieffel equivalence between ideals $I \triangleleft A$ and $J \triangleleft B$, namely, $I := \langle\!\langle \mathcal{E} \mid \mathcal{E}\rangle\!\rangle_A$ and $J := \langle \mathcal{E} \mid \mathcal{E}\rangle_B$.*

Proof. Let $\mathcal{E}$ be a Hilbert A, B-bimodule and let $I := \langle\!\langle \mathcal{E} \mid \mathcal{E}\rangle\!\rangle_A \triangleleft A$ and $J := \langle \mathcal{E} \mid \mathcal{E}\rangle_B \triangleleft B$. We may view $\mathcal{E}$ as a full Hilbert J-module, and $\alpha|_I$ is an isomorphism onto $\mathbb{K}(\mathcal{E})$ by Proposition 7.2. Thus $\mathcal{E}$ is a Morita–Rieffel equivalence from J to I by Theorem 6.4. Conversely, if $I \triangleleft A$ and $J \triangleleft B$ are ideals and $\mathcal{E}$ is a Morita–Rieffel equivalence between I and J, then the bimodule structure on $\mathcal{E}$ extends to an A, B-bimodule structure that gives a Hilbert A, B-bimodule by Corollary 3.13. □

Theorem 7.3 suggests to view Hilbert bimodules as *partial Morita–Rieffel equivalences.* This generalises the concept of a *partial automorphism* of a C*-algebra A, which is an isomorphism between two ideals in A (see [9]).

Theorem 7.4. *A correspondence is a Morita–Rieffel equivalence if and only if it comes from a Hilbert bimodule where both inner products are full, if and only if it comes from a Hilbert bimodule and is full, proper and faithful.*

Proof. We have seen at the beginning of this section that a Morita–Rieffel equivalence $\mathcal{E} \colon A \leftarrow B$ carries a left inner product. Conversely, let $\mathcal{E} \colon A \leftarrow B$ be a Hilbert bimodule and assume that the left and right inner products are full. Then $A = \langle\!\langle \mathcal{E} \mid \mathcal{E}\rangle\!\rangle_A$ acts through an isomorphism onto $\mathbb{K}(\mathcal{E})$ by Proposition 7.2. Thus $\mathcal{E}$ is an equivalence by Theorem 6.4.

Let $\mathcal{E} \colon A \leftarrow B$ be a Hilbert bimodule and $I := \langle\!\langle \mathcal{E} \mid \mathcal{E}\rangle\!\rangle_A$, $J := \langle \mathcal{E} \mid \mathcal{E}\rangle_B$. Being an equivalence says that $J = B$ and $I = A$. The condition $J = B$ says that $\mathcal{E}$ is full. The left action is through an isomorphism $I \xrightarrow{\simeq} \mathbb{K}(\mathcal{E})$. Being proper means that $\alpha(A) \subseteq \alpha(I) = \mathbb{K}(\mathcal{E})$. If α is also faithful, then this implies $A = I$. Conversely, if $I = A$, then $\mathcal{E}$ is proper and faithful. □

Theorem 7.5. *Let $\mathcal{E} \colon A \leftarrow B$ be a Hilbert bimodule. The following classes of objects form isomorphic lattices:*

(1) *ideals in A contained in $\langle\!\langle \mathcal{E} \mid \mathcal{E}\rangle\!\rangle_A$;*
(2) *ideals in B contained in $\langle \mathcal{E} \mid \mathcal{E}\rangle_B$;*

(3) *Hilbert A,B-subbimodules of $\mathcal{E}$.*
The bijections map $I \triangleleft \langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle_A$ and $J \triangleleft \langle \mathcal{E} \,|\, \mathcal{E} \rangle_B$ to the Hilbert subbimodules $I \cdot \mathcal{E}$ and $\mathcal{E} \cdot J$, respectively, and a Hilbert subbimodule $\mathcal{F}$ of $\mathcal{E}$ to the ideals $\langle\langle \mathcal{F} \,|\, \mathcal{F} \rangle\rangle_A$ and $\langle \mathcal{F} \,|\, \mathcal{F} \rangle_B$, respectively. Thus $\mathcal{F} \subseteq \mathcal{E}$ is a Morita–Rieffel equivalence between the ideals in A and B associated to it.

Proof. Let $\mathcal{F} \subseteq \mathcal{E}$ be a Hilbert subbimodule. We define ideals $\mathrm{r}(\mathcal{F}) := \langle\langle \mathcal{F} \,|\, \mathcal{F} \rangle\rangle_A \triangleleft \langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle_A$ and $\mathrm{s}(\mathcal{F}) := \langle \mathcal{F} \,|\, \mathcal{F} \rangle_B \triangleleft \langle \mathcal{E} \,|\, \mathcal{E} \rangle_B$. Then $\mathcal{F}$ is a Morita–Rieffel equivalence between $\mathrm{r}(\mathcal{F})$ and $\mathrm{s}(\mathcal{F})$ by Theorem 7.4. Conversely, let $I \triangleleft \langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle$ be an ideal. Then $\mathcal{E}_I := I \cdot \mathcal{E}$ is a Hilbert subbimodule with $\mathrm{r}(\mathcal{E}_I) = I^* \langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle_A I = I$. If $\mathcal{F}$ is another Hilbert subbimodule with $\mathrm{r}(\mathcal{F}) = I$, then

$$I \cdot \mathcal{E} = \langle\langle \mathcal{F} \,|\, \mathcal{F} \rangle\rangle_A \cdot \mathcal{E} = \mathcal{F} \cdot \langle \mathcal{F} \,|\, \mathcal{E} \rangle_B \subseteq \mathcal{F} = I \cdot \mathcal{F} \subseteq I \cdot \mathcal{E}.$$

Hence $I \cdot \mathcal{E}$ is the unique Hilbert subbimodule with $\mathrm{r}(\mathcal{F}) = I$. Thus r is bijective. A symmetric argument shows that s is bijective with inverse $J \mapsto \mathcal{E} \cdot J$. All our constructions preserve inclusions, so they are lattice isomorphisms. □

Corollary 7.6. *Morita–Rieffel equivalent* C*-*algebras have isomorphic ideal lattices.*

Proof. A Morita–Rieffel equivalence is a Hilbert bimodule with $\langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle_A = A$ and $\langle \mathcal{E} \,|\, \mathcal{E} \rangle_B = B$. Hence the isomorphism between the first two items in Theorem 7.5 is a lattice isomorphism $\mathbb{I}(A) \cong \mathbb{I}(B)$. □

Corollary 7.7. *Morita–Rieffel equivalent* C*-*algebras have isomorphic centres.*

Proof. This follows from the isomorphism of ideal lattices and the Dauns–Hofmann Theorem. □

Corollary 7.8. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert A,B-bimodules and let $i\colon \mathcal{E} \hookrightarrow \mathcal{F}$ be a correspondence map. Then $\mathcal{F} \cdot \langle \mathcal{E} \,|\, \mathcal{E} \rangle_B = \langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle_A \cdot \mathcal{F}$ and i is a unitary operator from $\mathcal{E}$ onto this Hilbert submodule of $\mathcal{F}$. The map i is an isomorphism if and only if $\langle \mathcal{E} \,|\, \mathcal{E} \rangle_B = \langle \mathcal{F} \,|\, \mathcal{F} \rangle_B$ if and only if $\langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle_A = \langle\langle \mathcal{F} \,|\, \mathcal{F} \rangle\rangle_A$.*

Proof. The image of i is a Hilbert subbimodule in $\mathcal{F}$. Theorem 7.5 shows that $i(\mathcal{E}) = I \cdot \mathcal{F} = \mathcal{F} \cdot J$ with $I := \langle\langle i(\mathcal{E}) \,|\, i(\mathcal{E}) \rangle\rangle_A = \langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle_A$ and $J := \langle i(\mathcal{E}) \,|\, i(\mathcal{E}) \rangle_B = \langle \mathcal{E} \,|\, \mathcal{E} \rangle_B$. Since a surjective isometry is unitary, i is unitary onto this Hilbert subbimodule. And $i(\mathcal{E}) = \mathcal{F}$ if and only if $\langle \mathcal{E} \,|\, \mathcal{E} \rangle_B = \langle \mathcal{F} \,|\, \mathcal{F} \rangle_B$, if and only if $\langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle_A = \langle\langle \mathcal{F} \,|\, \mathcal{F} \rangle\rangle_A$. □

Corollary 7.9. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert A,B- and B,C-bimodules. Then the correspondence $\mathcal{E} \otimes_B \mathcal{F}$ has a unique Hilbert A,C-bimodule structure.*

Proof. By Theorem 7.3, $\mathcal{E}$ is an equivalence from $J_1 := \langle \mathcal{E} \,|\, \mathcal{E} \rangle_B \triangleleft B$ to $I := \langle\langle \mathcal{E} \,|\, \mathcal{E} \rangle\rangle_A \triangleleft A$, and $\mathcal{F}$ is one from $K := \langle \mathcal{F} \,|\, \mathcal{F} \rangle_C \triangleleft C$ to $J_2 := \langle\langle \mathcal{F} \,|\, \mathcal{F} \rangle\rangle_B \triangleleft B$. Then $\mathcal{F} = J_2 \cdot \mathcal{F} \cong J_2 \otimes_B \mathcal{F}$ and $\mathcal{E} = \mathcal{E} \cdot J_1 = \mathcal{E} \otimes_B J_1$ by Proposition 3.12. Hence

$$\begin{aligned}\mathcal{E} \otimes_B \mathcal{F} \cong \mathcal{E} \otimes_B J_1 \otimes_B J_2 \otimes_B \mathcal{F} = \mathcal{E} \otimes_B (J_1 \cdot J_2) \otimes_B \mathcal{F} \\ = \mathcal{E} \cdot (J_1 J_2) \otimes_B (J_1 J_2) \cdot \mathcal{F}.\end{aligned}$$

The subspaces $\mathcal{E} \cdot (J_1 J_2)$ and $(J_1 J_2) \cdot \mathcal{F}$ are Morita–Rieffel equivalences from $J_1 J_2$ to I' and from K' to $J_1 J_2$ for some ideals $I' \triangleleft I$ and $K' \triangleleft K$ by Theorem 7.5. Hence $\mathcal{E} \otimes_B \mathcal{F}$ is a Morita–Rieffel equivalence from $K' \triangleleft C$ to $I' \triangleleft A$ and thus comes from a Hilbert A,C-bimodule by Theorem 7.3. □

Hence the Hilbert bimodules as arrows also give a subbicategory of $\mathfrak{Corr}$, which we denote by $\mathfrak{Corr}^*$. By Proposition 7.2, correspondence embeddings and Hilbert bimodule embeddings between two Hilbert bimodules are the same.

We have found four independent properties of correspondences that each define a subbicategory, namely, being proper, full, faithful or a Hilbert bimodule. The

Morita–Rieffel equivalences are exactly those correspondences with all four properties by Theorem 7.4.

Definition 7.10. Let $\widehat{A}$ for a C*-algebra denote the set of unitary equivalence classes of irreducible representations of A. If $I \in \mathbb{I}(A)$, let $\widehat{I} := \{[\pi] \in \widehat{A} : \pi|_I \neq 0\}$. There is a topology on $\widehat{A}$ whose open subsets are exactly those of the form $\widehat{I}$ for $I \in \mathbb{I}(A)$, and the map $I \mapsto \widehat{I}$ is a lattice isomorphism from $\mathbb{I}(A)$ onto the open subsets of $\widehat{A}$.

Theorem 7.11. *A Morita–Rieffel equivalence $\mathcal{E}$ between two* C^**-algebras A and B induces a homeomorphism $\widehat{\mathcal{E}}\colon \widehat{B} \to \widehat{A}$. It maps $[\pi] \in \widehat{B}$ to the unitary equivalence class of the induced representation of A on $\mathcal{E} \otimes_B \pi$.*

Proof. Any A,B-correspondence $\mathcal{E}$ induces a functor $\mathcal{E} \otimes_B \sqcup\colon \mathrm{Rep}(B) \to \mathrm{Rep}(A)$. By general bicategory theory, this functor is an equivalence if $\mathcal{E}$ is an equivalence. Therefore, a Morita–Rieffel equivalence maps irreducible representations to irreducible representations and preserves unitary equivalence classes of irreducible representations. Thus it induces a bijection $\widehat{B} \to \widehat{A}$. This is a homeomorphism for the topology defined above by Corollary 7.6. □

Corollary 7.12. *A Morita–Rieffel equivalence $\mathcal{E}$ between two* C^**-algebras A and B induces a homeomorphism* $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$.

Proof. $\mathrm{Prim}(A)$ is the T_0-quotient of $\widehat{A}$. That is, it is the quotient by the equivalence relation that identifies points that cannot be distinguished by open subsets. □

Example 7.13. Morita–Rieffel equivalence is not compatible with adding unit elements. We provide an example of Morita–Rieffel equivalent C*-algebras A and B such that A^+ and B^+ are not Morita–Rieffel equivalent.

Let $A = \mathbb{C}$, $B = \mathbb{K}(\ell^2\mathbb{N})$. These are Morita–Rieffel equivalent via $\mathcal{E} = \ell^2\mathbb{N}$. Since $A^+ \cong \mathbb{C} \oplus \mathbb{C}$ and $B^+ = \mathbb{K}(\ell^2\mathbb{N})^+ \subseteq \mathbb{B}(\ell^2\mathbb{N})$ have different centres, they cannot be Morita–Rieffel equivalent.

Theorem 7.14. *Let $\mathcal{E}$ be a Hilbert module over a* C^**-algebra B. Let $\mathcal{F} \subsetneq \mathcal{E}$ be a proper closed Hilbert submodule. There is an irreducible Hilbert space representation $\varrho\colon B \to \mathbb{B}(\mathcal{H})$ with $\mathcal{F} \otimes_\varrho \mathcal{H} \subsetneq \mathcal{E} \otimes_\varrho \mathcal{H}$.*

Proof. Let $K := \mathbb{K}(\mathcal{E})$ and let $R := \mathbb{K}(\mathcal{E}, \mathcal{F})$. This is a proper right ideal in K because $\mathcal{F} \subsetneq \mathcal{E}$. By [7, Theorem 2.9.5], there is a pure state ω on K that vanishes on R. Let $\varrho'\colon K \to \mathbb{B}(\mathcal{H}')$ be the irreducible GNS-representation of K associated to ω. Then $\varrho'(R)(\mathcal{H}') \neq \mathcal{H}'$ because every element of $\varrho'(R)(\mathcal{H}')$ is orthogonal to the cyclic vector of the GNS-representation by construction.

Let $I := \langle \mathcal{E} \,|\, \mathcal{E} \rangle \triangleleft B$. Thus $\mathcal{E}$ is a Morita–Rieffel equivalence between K and I. By Theorem 7.11, $[\varphi] \mapsto [\mathcal{E} \otimes_I \varphi]$ is a homeomorphism from $\widehat{I}$ to $\widehat{K}$. So there is an irreducible representation φ of I with $[\mathcal{E} \otimes_I \varphi] = [\varrho']$. Since φ is nondegenerate, it extends uniquely to a representation ϱ of B, which is still irreducible. And $\mathcal{E} \otimes_\varrho \mathcal{H} = \mathcal{E} \otimes_\varphi \mathcal{H} = \varrho'$ and $\mathcal{F} \otimes_\varrho \mathcal{H} = \mathcal{F} \otimes_\varphi \mathcal{H}$. We claim that $\mathcal{F} \otimes_\varrho \mathcal{H} \subsetneq \mathcal{E} \otimes_\varrho \mathcal{H}$ for this representation.

The operator $|\xi\rangle\langle\eta|$ for $\xi, \eta \in \mathcal{E}$ acts on $\mathcal{E} \otimes_\varrho \mathcal{H}$ by

$$\zeta \otimes \tau \mapsto \xi\langle \eta \,|\, \zeta \rangle \otimes \tau$$

for $\zeta \in \mathcal{E}$, $\tau \in \mathcal{H}$. Since $\mathbb{K}(\mathcal{F}, \mathcal{F})$ acts nondegenerately on $\mathcal{F}$, we have $R(\mathcal{E}) = \mathcal{F}$. Hence $R \subseteq K$ maps $\mathcal{E} \otimes_\varrho \mathcal{H}$ onto the subspace $\mathcal{F} \otimes_\varrho \mathcal{H}$. By construction of $\varrho' \cong \mathcal{E} \otimes_\varrho \mathcal{H}$, this is not the whole space. So $\mathcal{F} \otimes_\varrho \mathcal{H} \subsetneq \mathcal{E} \otimes_\varrho \mathcal{H}$. □

Any irreducible representation of B is unitarily equivalent to the GNS-representation for a pure state; so we may as well assume that the representation ϱ above comes from a pure state.

Remark 7.15. Theorem 7.14 was proven by [23, Proposition 1.16]. Kaad and Lesch [11] proved a weaker result and formulated Theorem 7.14 as a conjecture. They acknowledge the previous work of Pierrot in [12].

Exercise 7.16. *The left and right inner products on a Hilbert A,B-bimodule $\mathcal{E}$ induce the same norm, that is, $\|\langle\langle x \,|\, x\rangle\rangle_A\| = \|\langle x \,|\, x\rangle_B\|$ for all $x \in \mathcal{E}$.* (*Use Theorem* 3.16.)

Exercise 7.17. *Show that $\mathbb{K}(\mathcal{E}_1, \mathcal{E}_2)$ for two Hilbert B-modules $\mathcal{E}_1$ and $\mathcal{E}_2$ is a Hilbert $\mathbb{K}(\mathcal{E}_2), \mathbb{K}(\mathcal{E}_1)$-bimodule; use left and right composition, the left inner product $\langle\langle x_1 \,|\, x_2\rangle\rangle := x_1 x_2^*$, and the right inner product $\langle x_1 \,|\, x_2\rangle := x_1^* x_2$. This is a Morita–Rieffel equivalence between $\mathbb{K}(\mathcal{E}_1)$ and $\mathbb{K}(\mathcal{E}_2)$ if and only if $\langle \mathcal{E}_1 \,|\, \mathcal{E}_1\rangle_B = \langle \mathcal{E}_2 \,|\, \mathcal{E}_2\rangle_B$.*

Exercise 7.18. *Let A and B be* C*-*algebras. Let $\mathcal{E}$ be a Hilbert A,B-bimodule with left inner product $\langle\langle \xi \,|\, \eta\rangle\rangle_A$ and right inner product $\langle \xi \,|\, \eta\rangle_B$. Show that there is a* C*-*algebra with underlying vector space $A \oplus B \oplus \mathcal{E} \oplus \mathcal{E}^*$, multiplication*

$$(a_1, b_1, \xi_1, \eta_1) \cdot (a_2, b_2, \xi_2, \eta_2) \\ := (a_1 a_2 + \langle\langle \xi_1 \,|\, \eta_2\rangle\rangle_A, b_1 b_2 + \langle \eta_1 \,|\, \xi_2\rangle_B, a_1\xi_2 + \xi_1 b_2, a_2^*\eta_1 + \eta_2 b_1^*)$$

and involution $(a, b, \xi, \eta)^* := (a^*, b^*, \eta, \xi)$ for $a_i \in A$, $b_i \in B$ $\xi_i, \eta_i \in \mathcal{E} = \mathcal{E}^*$. This C*-*algebra L is called* linking algebra. *Show that $P(a, b, \xi, \eta) := (a, 0, \xi, 0)$ is a projection in $\mathcal{M}(L)$ with $A = PLP$ and $B = P^\perp L P^\perp$ for $P^\perp = 1 - P$. That is, A and B are complementary corners in L. When are P and $P^\perp$ full?*

Exercise 7.19. *Let X and Y be locally compact spaces. Describe a Hilbert module $\mathcal{E}$ over $\mathrm{C}_0(Y)$ through a continuous field $(\mathcal{H}_y)_{y\in Y}$ of Hilbert spaces as in Exercise* 1.13.

(1) *Show that $\mathbb{K}(\mathcal{E})$ is commutative if and only if all fibres $\mathcal{H}_y$ are at most* 1-*dimensional.*
(2) *Show that a continuous field of Hilbert spaces with finite-dimensional fibres is locally trivial, hence a vector bundle.*
(3) *Conclude that if $\mathcal{E}$ is a full Hilbert $\mathrm{C}_0(X), \mathrm{C}_0(Y)$-bimodule, then the underlying right Hilbert module is the space $\Gamma_0(Y, L)$ of C_0-sections of a* 1-*dimensional vector bundle $L \twoheadrightarrow Y$; such a vector bundle is called a* complex line bundle.
(4) *Describe a full Hilbert $\mathrm{C}_0(X), \mathrm{C}_0(Y)$-bimodule as $\Gamma_0(Y, L)$ as in* (3). *Show that the left action of $\mathrm{C}_0(X)$ on $\Gamma_0(Y, L)$ must be of the form $(f \cdot s)(y) = f(\varphi(y)) \cdot s(y)$ for a unique homeomorphism $\varphi \colon Y \xrightarrow{\simeq} X$.*
(5) *Conversely, any pair (L, φ) where L is a complex line bundle over Y and $\varphi \colon Y \xrightarrow{\simeq} X$ is a homeomorphism defines a full Hilbert bimodule over $\mathrm{C}_0(X)$ and $\mathrm{C}_0(Y)$.*
(6) *Show that the full Hilbert bimodules associated to pairs (L_1, φ_1) and (L_2, φ_2) are isomorphic if and only if $L_1 \cong L_2$ as vector bundles and $\varphi_1 = \varphi_2$. And the Hilbert bimodule isomorphisms are in bijection with the vector bundle isomorphisms $L_1 \cong L_2$.*

8. Hilbert bimodules form a higher inverse category

The partial *-isomorphisms between C*-algebras form an *inverse category* in the following sense:

Definition 8.1 ([14])**.** An *inverse category* is a category $\mathcal{C}$ such that for each $t \in \mathcal{C}$ there is a unique $u \in \mathcal{C}$ with $tut = t$ and $utu = u$; this unique element is denoted t^*. An endomorphism e of $\mathcal{C}$ is *idempotent* if $e^2 = e$. Let $E(\mathcal{C}) = \{e \in \mathcal{C} : e^2 = e\}$.

We are going to exhibit similar properties for the bicategory of Hilbert bimodules $\mathfrak{Corr}^*$. First we briefly recall some basic properties and examples of inverse categries and their actions (see also [17]).

Example 8.2. Let X be a topological space. A *partial homeomorphism* of X is a homeomorphism $f\colon U \xrightarrow{\cong} V$ between two open subsets $U, V \subseteq X$. We compose partial maps by $f \circ g(x) := f(g(x))$ for all $x \in X$ such that $g(x)$ and $f(g(x))$ are defined. This gives a partial homeomorphism if f and g are partial homeomorphisms. A partial homeomorphism $f\colon U \to V$ has a partial inverse $f^*\colon V \to U$. Inspection shows that this is the unique partial homeomorphism on X with $ff^*f = f$ and $f^*ff^* = f^*$. Thus the partial homeomorphisms of X form an inverse semigroup, which we denote by $I(X)$. An *inverse semigroup action* on X is a homomorphism from an inverse semigroup to $I(X)$.

A partial homeomorphism on X is idempotent if and only if it is the identity map on some open subset of X. The composite of two such partial identity maps is the identity map on the intersection of their domains. Thus the idempotent elements in $I(X)$ form a commutative subsemigroup, which is isomorphic to the set $\mathbb{O}(X)$ of open subsets of X with the associative multiplication $\cap$.

Example 8.3. Let A and B be C*-algebras. A *partial isomorphism* from A to B is a *-isomorphism between ideals in A and B. Partial isomorphisms with the composition of partial maps form an inverse category.

Theorem 8.4. *Let $\mathcal{C}$ be an inverse category. Then there is a family of sets X_y for $y \in \mathcal{C}$ and a faithful action of $\mathcal{C}$ by partial bijections between these. This action maps each arrow $t\colon y \to z$ in $\mathcal{C}$ to a partial bijection $t_*\colon X_y \to X_z$ in a functorial way, such that $t_* = u_*$ implies $t = u$. In particular, any inverse semigroup embeds into the inverse semigroup of partial bijections of some set.*

The theorem implies that inverse categories must inherit various algebraic properties of $I(X)$. In fact, the proof consists mostly in showing that any inverse category has these properties:

Proposition 8.5. *Let $\mathcal{C}$ be an inverse category.*
1. *$(t^*)^* = t$ for all $t \in \mathcal{C}$.*
2. *Idempotent endomorphisms in $\mathcal{C}$ are self-adjoint and form a commutative subcategory.*
3. *$(tu)^* = u^*t^*$ for all $t, u \in \mathcal{C}$.*
4. *If $e \in E(\mathcal{C})$ and $t \in \mathcal{C}$ are composable, then $tet^* \in E(\mathcal{C})$.*

Proof. Statement (1) holds because t satisfies the conditions $tt^*t = t$ and $t^*tt^* = t^*$, and these characterise $(t^*)^*$ uniquely. If $e \in E(\mathcal{C})$, then $eee = e$, and this uniquely determines e^*; thus $e = e^*$.

Let $e, f \in E(\mathcal{C})$ be composable. There is a unique $g \in \mathcal{C}$ with $(ef)g(ef) = ef$ and $g(ef)g = g$, namely, $g = (ef)^*$; then also $(ef)(fge)(ef) = ef$ and $(fge)(ef)(fge) = fge$; so $f(ef)^*e = (ef)^*$ by the uniqueness of the adjoint in $\mathcal{C}$. Then $(ef)^*(ef)^* = f(ef)^*ef(ef)^*e = f(ef)^*e = (ef)^*$, that is, $(ef)^*$ is also idempotent. Hence $(ef)^* = ef$. So ef is idempotent. By symmetry, fe is idempotent as well. Now $(ef)(fe)(ef) = efef = ef$ and $(fe)(ef)(fe) = fefe = fe$. Then $fe = (ef)^* = ef$. This finishes the proof of (2).

Let $e \in E(\mathcal{C})$ and $t \in \mathcal{C}$ be composable. Then $tet^*tet^* = tt^*tet^* = tet^*$ by commuting the idempotent elements e and tt^*. This proves (4). If $t, u \in \mathcal{C}$, then

$t^*u^*utt^*u^* = t^*tt^*u^*uu^* = t^*u^*$ and $utt^*u^*ut = uu^*u^*tt^*t = ut$ by commuting the idempotents u^*u and tt^*. Thus $t^*u^* = (ut)^*$, proving (3). □

Proof of Theorem 8.4. This is proved like Cayley's Theorem, by constructing a faithful action of an inverse category $\mathcal{C}$ on its arrow set by partial bijections. Let $\mathcal{C}^y := \{t \in \mathcal{C} : \mathrm{r}(t) = y\}$ for $y \in \mathcal{C}^0$. Each $t \in \mathcal{C}$ yields a map $\mathcal{C}^{\mathrm{s}(t)} \to \mathcal{C}^{\mathrm{r}(t)}$, $u \mapsto t \cdot u$. We will restrict these maps to partial bijections as required for an inverse semigroup action. The main issue is to find the right domain

$$D_t := \{t^*tu : u \in \mathcal{C}^{\mathrm{s}(t)}\} \subseteq \mathcal{C}^{\mathrm{s}(t)}.$$

Then $u \in D_t$ if and only if $t^*tu = u$ because $t^*t \in E(\mathcal{C})$. We claim that the left multiplication map $t \mapsto tu$ restricts to a bijection $\alpha_t : D_t \to D_{t^*}$ with inverse $\alpha_{t^*} : D_{t^*} \to D_t$. We compute $tu = tt^*tu \in D_{t^*}$ for all $u \in D_t$. So $\alpha_t(D_t) \subseteq D_{t^*}$. And also $\alpha_{t^*}\alpha_t(u) = t^*tu = u$ for all $u \in D_t$. By symmetry, $\alpha_{t^*}(D_{t^*}) \subseteq D_t$ and $\alpha_t\alpha_{t^*}(u) = u$ for all $u \in D_{t^*}$. Thus α_t is a partial homeomorphism with partial inverse α_{t^*} as asserted.

Let $t, u \in \mathcal{C}$. We claim that $\alpha_t \circ \alpha_u = \alpha_{tu}$. It is clear that $\alpha_t\alpha_u(v) = tuv = \alpha_{tu}(v)$ whenever both sides are defined. An element $v \in \mathcal{C}$ belongs to the domain of $\alpha_t\alpha_u$ if and only if $u^*uv = v$ and $t^*t(uv) = uv$; it belongs to the domain of α_{tu} if and only if $v = (tu)^*(tu)v$. If $u^*uv = v$ and $t^*t(uv) = uv$, then $(tu)^*(tu)v = u^*t^*t(uv) = u^*uv = v$ by Proposition 8.5.(2). Conversely, if $v = (tu)^*(tu)v$, then $u^*uv = u^*uu^*t^*tuv = u^*t^*tuv = v$ and $t^*tuv = t^*tuu^*t^*tuv = t^*tt^*tuu^*uv = uv$ by Proposition 8.5.(2). □

Definition 8.6. Let $\mathcal{C}$ be an inverse category. If $t, u \in \mathcal{C}$, then we write $t \le u$ if $\mathrm{s}(t) = \mathrm{s}(u)$ and $t = ut^*t$.

Exercise 8.7. *Show that $t, u \in \mathcal{C}$ satisfy $t \le u$ if and only if $t^* \le u^*$, if and only if there is an idempotent element $e \in E(\mathcal{C})$ with $\mathrm{r}(e) = \mathrm{s}(u)$ and $t = u \cdot e$. Show that if $t, u, v, w \in \mathcal{C}$ satisfy $t \le u$, $v \le w$ and $\mathrm{s}(t) = \mathrm{s}(u) = \mathrm{r}(v) = \mathrm{r}(w)$, then $t \cdot v \le u \cdot w$. Show that $\le$ is a partial order on $\mathcal{C}$.*

Any Hilbert bimodule $\mathcal{E} : A \xleftarrow{\mathcal{C}} B$ has a canonical "adjoint" $\mathcal{E}^* : B \xleftarrow{\mathcal{C}} A$, namely, the conjugate vector space of $\mathcal{E}$ with left and right Hilbert module structures exchanged. We are going to explore the properties of this involution on the bicategory $\mathfrak{Corr}^*$ of Hilbert bimodules.

Theorem 8.8. *Let $\mathcal{E}$ be a Hilbert A,B-bimodule for two C^*-algebras A and B. There are natural Hilbert bimodule isomorphisms*

$$\mathcal{E} \otimes_B \mathcal{E}^* \cong \langle\langle \mathcal{E} \mid \mathcal{E} \rangle\rangle_A, \qquad \mathcal{E}^* \otimes_A \mathcal{E} \cong \langle \mathcal{E} \mid \mathcal{E} \rangle_B.$$

These induce isomorphisms

$$\mu : \mathcal{E} \otimes_B \mathcal{E}^* \otimes_A \mathcal{E} \cong \mathcal{E}, \qquad \mu^* : \mathcal{E}^* \otimes_A \mathcal{E} \otimes_B \mathcal{E}^* \cong \mathcal{E}^*$$

such that

$$\mu \otimes_B \mathrm{id}_{\mathcal{E}^*} = \mathrm{id}_{\mathcal{E}} \otimes_B \mu^* : \mathcal{E} \otimes_B \mathcal{E}^* \otimes_A \mathcal{E} \otimes_B \mathcal{E}^* \cong \mathcal{E} \otimes_B \mathcal{E}^*,$$
$$\mu^* \otimes_A \mathrm{id}_{\mathcal{E}} = \mathrm{id}_{\mathcal{E}^*} \otimes_A \mu : \mathcal{E}^* \otimes_A \mathcal{E} \otimes_B \mathcal{E}^* \otimes_A \mathcal{E} \cong \mathcal{E}^* \otimes_A \mathcal{E}.$$

Proof. By Theorem 7.3, $\mathcal{E}$ is a Morita–Rieffel equivalence between $I := \langle\langle \mathcal{E} \mid \mathcal{E} \rangle\rangle_A \triangleleft A$ and $J := \langle \mathcal{E} \mid \mathcal{E} \rangle_B \triangleleft B$. Theorem 6.4 says that the left and right inner products on $\mathcal{E}$ induce correspondence isomorphisms $\mu_{\mathcal{E}\mathcal{E}^*} : \mathcal{E} \otimes_B \mathcal{E}^* \xrightarrow{\simeq} I$ and $\mu_{\mathcal{E}^*\mathcal{E}} : \mathcal{E}^* \otimes_A \mathcal{E} \xrightarrow{\simeq} J$. The right and left multiplication maps induce isomorphisms $\mu_{\mathcal{E}I} : \mathcal{E} \otimes_B I \xrightarrow{\simeq} \mathcal{E}$ and

$\mu_{J\mathcal{E}}\colon J\otimes_A\mathcal{E}\xrightarrow{\simeq}\mathcal{E}$ by Proposition 3.12, and similarly for $\mathcal{E}^*$. This gives commuting diagrams of correspondence isomorphisms

$$\begin{tikzcd}
\mathcal{E}\otimes_B\mathcal{E}^*\otimes_A\mathcal{E} \arrow[r, Rightarrow, "\mathrm{id}_{\mathcal{E}}\otimes_B\mu_{\mathcal{E}^*\mathcal{E}}"] \arrow[d, Rightarrow, "\mu_{\mathcal{E}\mathcal{E}^*}\otimes_A\mathrm{id}_{\mathcal{E}}"] & \mathcal{E}\otimes_B J \arrow[d, Rightarrow, "\mu_{\mathcal{E}J}"] \\
I\otimes_A\mathcal{E} \arrow[r, Rightarrow, "\mu_{I\mathcal{E}}"'] & \mathcal{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}^*\otimes_A\mathcal{E}\otimes_B\mathcal{E}^* \arrow[r, Rightarrow, "\mathrm{id}_{\mathcal{E}^*}\otimes_A\mu_{\mathcal{E}\mathcal{E}^*}"] \arrow[d, Rightarrow, "\mu_{\mathcal{E}^*\mathcal{E}}\otimes_B\mathrm{id}_{\mathcal{E}^*}"] & \mathcal{E}^*\otimes_A I \arrow[d, Rightarrow, "\mu_{\mathcal{E}^*I}"] \\
J\otimes_B\mathcal{E}^* \arrow[r, Rightarrow, "\mu_{J\mathcal{E}^*}"'] & \mathcal{E}^*
\end{tikzcd}$$

because $x_1\cdot\langle x_2\,|\,x_3\rangle_B = \langle\!\langle x_1\,|\,x_2\rangle\!\rangle_A\cdot x_3$ for $x_1,x_3,x_3\in\mathcal{E}$, and similarly for $\mathcal{E}^*$, with the inner products $\langle x_1^*\,|\,x_2^*\rangle_A = \langle\!\langle x_1\,|\,x_2\rangle\!\rangle_A$ and $\langle\!\langle x_1^*\,|\,x_2^*\rangle\!\rangle_B = \langle x_1\,|\,x_2\rangle_B$ on $\mathcal{E}^*$. These are the isomorphisms μ and μ^* in the statement of the theorem.

The isomorphisms $\mu_{\mathcal{E}\mathcal{E}^*}\circ(\mu\otimes_B\mathrm{id}_{\mathcal{E}^*})$ and $\mu_{\mathcal{E}\mathcal{E}^*}\circ(\mathrm{id}_{\mathcal{E}}\otimes_B\mu^*)$ from $\mathcal{E}\otimes_B\mathcal{E}^*\otimes_A\mathcal{E}\otimes_B\mathcal{E}^*$ to I both map $x_1\otimes x_2^*\otimes x_3\otimes x_4^*$ to $\langle\!\langle x_1\,|\,x_2\rangle\!\rangle_A\cdot\langle\!\langle x_3\,|\,x_4\rangle\!\rangle_A$. Since $\mu_{\mathcal{E}\mathcal{E}^*}$ is an isomorphism, this implies $\mu\otimes_B\mathrm{id}_{\mathcal{E}^*} = \mathrm{id}_{\mathcal{E}}\otimes_B\mu^*$. Similarly, $\mu^*\otimes_A\mathrm{id}_{\mathcal{E}} = \mathrm{id}_{\mathcal{E}^*}\otimes_A\mu$. □

Theorem 8.9. *Let A,B,C be* C**-algebras. Let $\mathcal{E}\colon A\xleftarrow{c} B$ and $\mathcal{F}\colon A\xleftarrow{c} C$ be correspondences and let $\mathcal{G}$ be a Hilbert B,C-bimodule. If $\langle\mathcal{E}\,|\,\mathcal{E}\rangle_B\subseteq\langle\!\langle\mathcal{G}\,|\,\mathcal{G}\rangle\!\rangle_B$, then there is a natural bijection between the sets of* 2*-arrows $\mathcal{E}\otimes_B\mathcal{G}\to\mathcal{F}$ and $\mathcal{E}\to\mathcal{F}\otimes_C\mathcal{G}^*$. This preserves invertibility of* 2*-arrows if and only if $\langle\mathcal{F}\,|\,\mathcal{F}\rangle_C\subseteq\langle\mathcal{G}\,|\,\mathcal{G}\rangle_C$.*

Proof. Let $i\colon\mathcal{E}\otimes_B\mathcal{G}\hookrightarrow\mathcal{F}$ be a correspondence map. Then so is $i\otimes\mathrm{id}_{\mathcal{G}^*}\colon\mathcal{E}\otimes_B\mathcal{G}\otimes_C\mathcal{G}^*\hookrightarrow\mathcal{F}\otimes_B\mathcal{G}^*$. The left inner product induces an isomorphism $\mu_{\mathcal{G}\mathcal{G}^*}\colon\mathcal{G}\otimes_C\mathcal{G}^*\xrightarrow{\simeq}\langle\!\langle\mathcal{G}\,|\,\mathcal{G}\rangle\!\rangle_B$. If $\langle\mathcal{E}\,|\,\mathcal{E}\rangle_B\subseteq\langle\!\langle\mathcal{G}\,|\,\mathcal{G}\rangle\!\rangle_B$, then $\mathcal{E}$ is a nondegenerate right module over the ideal $\langle\!\langle\mathcal{G}\,|\,\mathcal{G}\rangle\!\rangle_B$, so that $\mathcal{E}\otimes_B\mathcal{G}\otimes_C\mathcal{G}^*\cong\mathcal{E}$ by the multiplication map $x\otimes y\otimes z^*\mapsto x\langle\!\langle y\,|\,z\rangle\!\rangle_B$. Thus i induces a correspondence map $i^\flat\colon\mathcal{E}\hookrightarrow\mathcal{F}\otimes_B\mathcal{G}^*$, defined by $i^\flat(x\cdot\langle\!\langle y\,|\,z\rangle\!\rangle_B) = i(x\otimes y)\otimes z^*$. If i is an isomorphism, so is $i^\flat$.

Conversely, let $j\colon\mathcal{E}\hookrightarrow\mathcal{F}\otimes_C\mathcal{G}^*$ be a correspondence map. Then so is $j\otimes_B\mathrm{id}_{\mathcal{G}}\colon\mathcal{E}\otimes_B\mathcal{G}\hookrightarrow\mathcal{F}\otimes_C\mathcal{G}^*\otimes_B\mathcal{G}$. The right inner product induces an isomorphism $\mu_{\mathcal{G}^*\mathcal{G}}\colon\mathcal{G}^*\otimes_B\mathcal{G}\xrightarrow{\simeq}\langle\mathcal{G}\,|\,\mathcal{G}\rangle_C\triangleleft C$. We get a correspondence map $\mathcal{F}\otimes_C\mathcal{G}^*\otimes_B\mathcal{G}\cong\mathcal{F}\cdot\langle\mathcal{G}\,|\,\mathcal{G}\rangle_C\hookrightarrow\mathcal{F}$, $x\otimes y^*\otimes z\mapsto x\langle y\,|\,z\rangle_B$, and j induces a correspondence map $j^\#\colon\mathcal{E}\otimes_B\mathcal{G}\hookrightarrow\mathcal{F}$. If j is an isomorphism, then $j^\#$ is one if and only if the embedding $\mathcal{F}\otimes_C\mathcal{G}^*\otimes_B\mathcal{G}\hookrightarrow\mathcal{F}$ is an isomorphism. As above, this happens if and only if $\langle\mathcal{F}\,|\,\mathcal{F}\rangle_C\subseteq\langle\mathcal{G}\,|\,\mathcal{G}\rangle_C$.

The following commuting diagram shows that $(i^\flat)^\# = i$:

$$\begin{tikzcd}
\mathcal{E}\otimes_B\mathcal{G}\otimes_C\mathcal{G}^*\otimes_B\mathcal{G} \arrow[r, Rightarrow, "i\otimes_C\mathrm{id}_{\mathcal{G}^*\otimes_B\mathcal{G}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\mathcal{E}}\otimes_B\mu"', "\cong"] & \mathcal{F}\otimes_C\mathcal{G}^*\otimes_B\mathcal{G} \arrow[d, Rightarrow, "\mathrm{id}_{\mathcal{F}}\otimes_C\mu_{\mathcal{G}^*,\mathcal{G}}"] \\
\mathcal{E}\otimes_B\mathcal{G} \arrow[r, Rightarrow, "i"'] \arrow[ur, Rightarrow, "i^\flat\otimes_B\mathrm{id}_{\mathcal{G}}"] & \mathcal{F}
\end{tikzcd}$$

Here $\mu\colon\mathcal{G}\otimes_C\mathcal{G}^*\otimes_B\mathcal{G}\to\mathcal{G}$ is as in Theorem 8.8. The square and the upper left triangle commute because $\mu(x_1\otimes x_2^*\otimes x_3) = x_1\langle x_2\,|\,x_3\rangle = \langle\!\langle x_1\,|\,x_2\rangle\!\rangle x_3$. Since the left vertical map is an isomorphism, this implies that the lower right triangle commutes; that is, $(i^\flat)^\# = i$. Similarly, the following diagram commutes and gives $(j^\#)^\flat = j$:

$$\begin{tikzcd}
\mathcal{E}\otimes_B\mathcal{G}\otimes_C\mathcal{G}^* \arrow[r, Rightarrow, "j\otimes_B\mathrm{id}_{\mathcal{G}\otimes_C\mathcal{G}^*}"] \arrow[d, Rightarrow, "\mathrm{id}_{\mathcal{E}}\otimes_B\mu_{\mathcal{G},\mathcal{G}^*}"', "\cong"] \arrow[dr, Rightarrow, "j^\#\otimes_C\mathrm{id}_{\mathcal{G}^*}"] & \mathcal{F}\otimes_C\mathcal{G}^*\otimes_B\mathcal{G}\otimes_C\mathcal{G}^* \arrow[d, Rightarrow, "\cong"', "\mathrm{id}_{\mathcal{F}}\otimes_C\mu"] \\
\mathcal{E} \arrow[r, Rightarrow, "j"'] & \mathcal{F}\otimes_C\mathcal{G}^*
\end{tikzcd}$$

Hence the constructions above are bijections inverse to each other. □

Theorem 8.10. *Let A,B,C be* C**-algebras. Let $\mathcal{E}\colon B\xleftarrow{c} C$ and $\mathcal{F}\colon A\xleftarrow{c} C$ be correspondences and let $\mathcal{G}$ be a Hilbert A,B-bimodule. If $\langle\mathcal{G}\,|\,\mathcal{G}\rangle_B\cdot\mathcal{E} = \mathcal{E}$, then there* *is a natural bijection between the sets of* 2*-arrows* $\mathcal{G} \otimes_B \mathcal{E} \to \mathcal{F}$ *and* $\mathcal{E} \to \mathcal{G}^* \otimes_B \mathcal{F}$. *This preserves invertibility of* 2*-arrows if and only if* $\langle\langle \mathcal{G} \mid \mathcal{G} \rangle\rangle_A \mathcal{F} = \mathcal{F}$.

Proof. Let $i\colon \mathcal{G} \otimes_B \mathcal{E} \hookrightarrow \mathcal{F}$ be a correspondence map. Then so is $\mathrm{id}_{\mathcal{G}^*} \otimes_A i\colon \mathcal{G}^* \otimes_A \mathcal{G} \otimes_B \mathcal{E} \hookrightarrow \mathcal{G}^* \otimes_A \mathcal{F}$. Since $\langle \mathcal{G} \mid \mathcal{G} \rangle \mathcal{E} = \mathcal{E}$, the map $\mathcal{G}^* \otimes_A \mathcal{G} \otimes_B \mathcal{E} \to \mathcal{E}$, $x_1^* \otimes x_2 \mapsto y \mapsto \langle x_1 \mid x_2 \rangle_B y$, is an isomorphism. Hence i induces a correspondence map $i^\flat\colon \mathcal{E} \hookrightarrow \mathcal{G}^* \otimes_A \mathcal{F}$, which is an isomorphism of correspondences if and only if i is one.

Conversely, let $j\colon \mathcal{E} \hookrightarrow \mathcal{G}^* \otimes_A \mathcal{F}$ be an embedding of correspondences. It induces an embedding of correspondences $\mathrm{id}_{\mathcal{G}} \otimes_A j\colon \mathcal{G} \otimes_A \mathcal{E} \hookrightarrow \mathcal{G} \otimes_A \mathcal{G}^* \otimes_A \mathcal{F}$. The map $x_1 \otimes x_2^* \otimes y \mapsto \langle\langle x_1 \mid x_2 \rangle\rangle_B y$ for $x_1, x_2 \in \mathcal{G}$, $y \in \mathcal{F}$ defines a correspondence embedding $\mathcal{G} \otimes_A \mathcal{G}^* \otimes_A \mathcal{F} \hookrightarrow \mathcal{F}$, which is an isomorphism if and only if $\langle\langle \mathcal{G} \mid \mathcal{G} \rangle\rangle_A \mathcal{F} = \mathcal{F}$. Thus j induces an embedding of correspondences $j^\#\colon \mathcal{G} \otimes_A \mathcal{E} \hookrightarrow \mathcal{F}$, which is an isomorphism if and only if j is an isomorphism and $\langle\langle \mathcal{G} \mid \mathcal{G} \rangle\rangle_A \mathcal{F} = \mathcal{F}$. Diagrams similar to those in the proof of Theorem 8.9 imply $(i^\flat)^\# = i$ and $(j^\#)^\flat = j$ for all i and j, so our two constructions are bijections inverse to each other. □

Theorem 8.11. *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be a Hilbert* A,B*-bimodule and a Hilbert* B,A*-bimodule with* $\mathcal{E} \otimes_B \mathcal{F} \otimes_A \mathcal{E} \cong \mathcal{E}$ *and* $\mathcal{F} \otimes_A \mathcal{E} \otimes_B \mathcal{F} \cong \mathcal{F}$. *There is a* unique *isomorphism* $\varphi\colon \mathcal{F} \cong \mathcal{E}^*$ *such that the composite map*

$$\mathcal{E} \cong \mathcal{E} \otimes_B \mathcal{F} \otimes_A \mathcal{E} \cong \mathcal{E} \otimes_B \mathcal{E}^* \otimes_A \mathcal{E} \cong \mathcal{E} \tag{8.1}$$

is the identity map.

In a bicategory, we cannot expect that for an arrow t there is a unique arrow t^* with $tt^*t \cong t$ and $t^*tt^* \cong t^*$. Theorem 8.11 asserts the next best thing: a canonical isomorphism between any two arrows t^* with this property.

Proof. Denote the natural isomorphisms in the statement by $\lambda\colon \mathcal{E} \otimes_B \mathcal{F} \otimes_A \mathcal{E} \xrightarrow{\cong} \mathcal{E}$ and $\lambda^*\colon \mathcal{F} \otimes_A \mathcal{E} \otimes_B \mathcal{F} \xrightarrow{\cong} \mathcal{F}$. There is a natural isomorphism

$$\begin{aligned} \langle \mathcal{E} \mid \mathcal{E} \rangle \cdot \mathcal{F} \cdot \langle\langle \mathcal{E} \mid \mathcal{E} \rangle\rangle &\cong \mathcal{E}^* \otimes_A \mathcal{E} \otimes_B \mathcal{F} \otimes_A \mathcal{E} \otimes_B \mathcal{E}^* \\ &\xrightarrow[\cong]{\mathrm{id}_{\mathcal{E}^*} \otimes_A \lambda \otimes_B \mathrm{id}_{\mathcal{E}^*}} \mathcal{E}^* \otimes_A \mathcal{E} \otimes_B \mathcal{E}^* \cong \mathcal{E}^*. \end{aligned} \tag{8.2}$$

Thus $\mathcal{E}^* \subseteq \mathcal{F}$. Then $\langle\langle \mathcal{E} \mid \mathcal{E} \rangle\rangle \subseteq \langle \mathcal{F} \mid \mathcal{F} \rangle$ and $\langle \mathcal{E} \mid \mathcal{E} \rangle \subseteq \langle\langle \mathcal{F} \mid \mathcal{F} \rangle\rangle$. The same argument with the other isomorphism $\mathcal{F} \otimes_A \mathcal{E} \otimes_B \mathcal{F} \cong \mathcal{F}$ gives $\langle\langle \mathcal{F} \mid \mathcal{F} \rangle\rangle \subseteq \langle \mathcal{E} \mid \mathcal{E} \rangle$ and $\langle \mathcal{F} \mid \mathcal{F} \rangle \subseteq \langle\langle \mathcal{E} \mid \mathcal{E} \rangle\rangle$. These four inclusions together say that $\langle\langle \mathcal{F} \mid \mathcal{F} \rangle\rangle = \langle \mathcal{E} \mid \mathcal{E} \rangle$ and $\langle \mathcal{F} \mid \mathcal{F} \rangle = \langle\langle \mathcal{E} \mid \mathcal{E} \rangle\rangle$. Thus $\langle \mathcal{E} \mid \mathcal{E} \rangle \cdot \mathcal{F} \cdot \langle\langle \mathcal{E} \mid \mathcal{E} \rangle\rangle = \mathcal{F}$, and (8.2) is an isomorphism $\mathcal{F} \cong \mathcal{E}^*$.

Theorem 8.9 and Theorem 8.10 give bijections between the sets of Hilbert bimodule isomorphisms

$$\mathcal{E} \otimes_B \mathcal{F} \otimes_A \mathcal{E} \cong \mathcal{E}, \qquad \mathcal{E} \otimes_B \mathcal{F} \cong \mathcal{E} \otimes_A \mathcal{E}^*, \qquad \mathcal{F} \cong \mathcal{E}^* \otimes_B \mathcal{E} \otimes_A \mathcal{E}^*.$$

Since $\mathcal{E}^* \otimes_B \mathcal{E} \otimes_A \mathcal{E}^* \cong \mathcal{E}^*$, there is a unique isomorphism $\mathcal{F} \xrightarrow{\cong} \mathcal{E}^*$ that corresponds to the given isomorphism $\mathcal{E} \otimes_B \mathcal{F} \otimes_A \mathcal{E} \xrightarrow{\cong} \mathcal{E}$. This is also the unique isomorphism for which the composite map in (8.1) is the identity map. (In fact, this unique isomorphism $\mathcal{F} \cong \mathcal{E}^*$ is the one constructed above.) □

Proposition 8.5 describes some important general properties of inverse semigroups. Properties (1) and (3) are obvious for Hilbert bimodules, but the other properties are not. The following theorem and exercises provide analogous statements for Hilbert bimodules. The mere existence of certain isomorphisms follows already because isomorphism classes of Hilbert bimodules form an inverse category. More work is needed, however, to pin down these isomorphisms.

Theorem 8.12. *Let $\mathcal{E}\colon A \underset{\mathcal{E}}{\leftarrow} A$ be a Hilbert bimodule with an isomorphism $\lambda\colon \mathcal{E} \otimes_A \mathcal{E} \cong \mathcal{E}$. There is a unique isomorphism from $\mathcal{E}$ onto the ideal $I := \langle \mathcal{E} \,|\, \mathcal{E}\rangle_A = \langle\!\langle \mathcal{E} \,|\, \mathcal{E}\rangle\!\rangle_A$ in A that intertwines λ and the multiplication map $I \otimes_A I \cong I$. This map is associative.*

If $\lambda_i\colon \mathcal{E}_i \otimes_A \mathcal{E}_i \to \mathcal{E}_i$ for $i = 1, 2$ are two Hilbert bimodule isomorphisms, then there is a unique isomorphism $\sigma\colon \mathcal{E}_1 \otimes_A \mathcal{E}_2 \xrightarrow{\cong} \mathcal{E}_2 \otimes_A \mathcal{E}_1$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{E}_1 \otimes_A \mathcal{E}_2 \otimes_A \mathcal{E}_1 \arrow[r, Rightarrow, "\mathrm{id}\otimes\sigma^{-1}"] \arrow[dr, Rightarrow, "\sigma\otimes\mathrm{id}"'] & \mathcal{E}_1 \otimes_A \mathcal{E}_1 \otimes_A \mathcal{E}_2 \arrow[r, Rightarrow, "\lambda_1\otimes\mathrm{id}"] & \mathcal{E}_1 \otimes_A \mathcal{E}_2 \arrow[d, Rightarrow, "\sigma"] \\
& \mathcal{E}_2 \otimes_A \mathcal{E}_1 \otimes_A \mathcal{E}_1 \arrow[r, Rightarrow, "\mathrm{id}\otimes\lambda_1"] & \mathcal{E}_2 \otimes_A \mathcal{E}_1
\end{tikzcd}$$

The same isomorphism σ also makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{E}_2 \otimes_A \mathcal{E}_1 \otimes_A \mathcal{E}_2 \arrow[r, Rightarrow, "\mathrm{id}\otimes\sigma"] \arrow[dr, Rightarrow, "\sigma^{-1}\otimes\mathrm{id}"'] & \mathcal{E}_2 \otimes_A \mathcal{E}_2 \otimes_A \mathcal{E}_1 \arrow[r, Rightarrow, "\lambda_2\otimes\mathrm{id}"] & \mathcal{E}_2 \otimes_A \mathcal{E}_1 \\
& \mathcal{E}_1 \otimes_A \mathcal{E}_2 \otimes_A \mathcal{E}_2 \arrow[r, Rightarrow, "\mathrm{id}\otimes\lambda_2"] & \mathcal{E}_1 \otimes_A \mathcal{E}_2 \arrow[u, Rightarrow, "\sigma"]
\end{tikzcd}$$

So $\mathcal{E}_1 \otimes_A \mathcal{E}_2 \otimes_A \mathcal{E}_1 \otimes_A \mathcal{E}_2 \cong \mathcal{E}_1 \otimes_A \mathcal{E}_2$.

Proof. The isomorphism λ induces an isomorphism $\mathcal{E} \otimes_A \mathcal{E} \otimes_A \mathcal{E} \cong \mathcal{E}$. Hence $\mathcal{E} = \mathcal{F}$ satisfies the conditions in Theorem 8.11 which characterise $\mathcal{E}^*$. This gives an isomorphism $\mathcal{E} \cong \mathcal{E}^*$. Hence λ gives isomorphisms

$$\langle\!\langle \mathcal{E} \,|\, \mathcal{E}\rangle\!\rangle_A \cong \mathcal{E}^* \otimes_A \mathcal{E} \cong \mathcal{E} \cong \mathcal{E} \otimes_A \mathcal{E}^* \cong \langle \mathcal{E} \,|\, \mathcal{E}\rangle_A.$$

If two ideals in A are isomorphic as right Hilbert modules (or correspondences), then they are conjugate by a unitary, hence equal. Thus $\langle \mathcal{E} \,|\, \mathcal{E}\rangle = \langle\!\langle \mathcal{E} \,|\, \mathcal{E}\rangle\!\rangle$, and $\mathcal{E}$ is isomorphic as a correspondence to this ideal equipped with the canonical Hilbert A-bimodule structure. So we may assume without loss of generality that $\mathcal{E} = I$ for an ideal I, with the canonical Hilbert bimodule structure.

The map $\lambda\colon I \otimes_A I \to I$ is of the form $\lambda(a_1 \otimes a_2) = ua_1a_2$ for some $u \in \mathcal{U}(I)$. Since λ is a bimodule map, u must be central. An isomorphism from I onto an ideal in I can only be of the form $I \to I$, $a \mapsto va$, for some central unitary multiplier v of I. The isomorphism coming from v intertwines λ above and the multiplication map on I if and only if $(va_1) \cdot (va_2) = vua_1a_2$ for all $a_1, a_2 \in I$. Since u and v are both central, this is equivalent to $u = v$. Hence this is the unique isomorphism on I that intertwines λ and the usual multiplication on A.

Since $\langle\!\langle I \,|\, I\rangle\!\rangle_A = I = \langle I \,|\, I\rangle_A$ and the multiplication $I \otimes_A I \to I$ is associative, the same must hold for $\mathcal{E}$ and λ; that is, $\lambda \circ (\mathrm{id}_{\mathcal{E}} \otimes \lambda) = \lambda \circ (\lambda \otimes \mathrm{id}_{\mathcal{E}})$.

Let $\lambda_i\colon \mathcal{E}_i \otimes_A \mathcal{E}_i \xrightarrow{\cong} \mathcal{E}_i$ for $i = 1, 2$ be as above. Identify $\mathcal{E}_i \cong J_i$ for ideals $J_i \triangleleft A$ with the canonical bimodule structure, so that λ_i becomes the multiplication map. The multiplication map in A is a Hilbert bimodule isomorphism $A \otimes_A A \cong A$. On the Hilbert subbimodule $J_1 \otimes_A J_2$, this restricts to an isometric embedding into A. By Corollary 7.8, this is an isomorphism onto $J_1 \cdot \langle J_2 \,|\, J_2\rangle_A = J_1 \cdot J_2 = J_1 \cap J_2$. Similarly, $J_2 \otimes_A J_1 \cong J_1 \cap J_2$. This gives an isomorphism $\mathcal{E}_1 \otimes_A \mathcal{E}_2 \cong \mathcal{E}_2 \otimes_A \mathcal{E}_1$. It makes the two diagrams in the theorem commute.

Another isomorphism $\sigma\colon J_1 \otimes_A J_2 \xrightarrow{\cong} J_2 \otimes_A J_1$ must be of the form $\sigma'(x) = u \cdot x$ for a central unitary multipler u of $J_1 \cap J_2 \triangleleft A$. This unitary also commutes with the images of $\mathcal{M}(A)$ and $\mathcal{M}(J_i)$ for $i = 1, 2$. In the two diagrams above for σ', one isomorphism contains the unitary uu^*, the other only u or only u^*. Hence either diagram for σ' commutes if and only if $u = 1$. Thus σ is unique as desired. $\square$

Exercise 8.13. *Let $\mathcal{E}$ be a Hilbert A-bimodule and let $\lambda\colon \mathcal{E}\otimes_A\mathcal{E} \xrightarrow{\cong} \mathcal{E}$ be an isomorphism. There is a unique isomorphism $\mathcal{E}^*\cong\mathcal{E}$ so that the composite isomorphism*

$$\mathcal{E}\otimes_A\mathcal{E}\otimes_A\mathcal{E}\cong\mathcal{E}\otimes_A\mathcal{E}^*\otimes_A\mathcal{E}\xrightarrow[\mu]{\cong}\mathcal{E}$$

is $\lambda\circ(\mathrm{id}_{\mathcal{E}}\otimes\lambda)=\lambda\circ(\lambda\otimes\mathrm{id}_{\mathcal{E}})$.

Exercise 8.14. *Let $\mathcal{F}$ be a Hilbert A,B-bimodule, $\mathcal{E}$ a Hilbert A-bimodule, and $\mathcal{G}:=\mathcal{F}^*\otimes_A\mathcal{E}\otimes_A\mathcal{F}$. Given an isomorphism $\lambda\colon\mathcal{E}\otimes_A\mathcal{E}\xrightarrow{\cong}\mathcal{E}$, construct a canonical isomorphism $\mathcal{G}\otimes_B\mathcal{G}\xrightarrow{\cong}\mathcal{G}$.*

Theorem 8.11 does not characterise Hilbert bimodules among correspondences. That is, there are correspondences $\mathcal{E}\colon A\xleftarrow{\mathcal{C}} B$ for which there is $\mathcal{F}$ with $\mathcal{E}\otimes_B\mathcal{F}\otimes_A\mathcal{E}\cong\mathcal{E}$ and $\mathcal{F}\otimes_A\mathcal{E}\otimes_B\mathcal{F}\cong\mathcal{F}$, but which are not Hilbert bimodules:

Example 8.15. Let $A=\mathbb{K}(\ell^2\mathbb{N})^+$ and $B=\mathbb{C}$, let $u\colon B\to A$ be the unit map and $\varepsilon\colon A\to B$ the augmentation map. These are unital *-homomorphisms, which we may view as correspondences. Since $\varepsilon\circ u=\mathrm{id}_B$, we have $\varepsilon\circ u\circ\varepsilon=\varepsilon$ and $u\circ\varepsilon\circ u=u$. We claim that ε is the only correspondence $A\xleftarrow{\mathcal{C}} B$ with this property, so that we have found the desired counterexample.

A correspondence $\mathcal{E}\colon A\xleftarrow{\mathcal{C}} B$ is a Hilbert space $\mathcal{H}$ with a unital representation ϱ of A. A correspondence $B\xleftarrow{\mathcal{C}} A$ is simply a Hilbert A-module. An isomorphism $u\otimes_A\mathcal{E}\otimes_{\mathbb{C}}u\cong u$ of correspondences $B\xleftarrow{\mathcal{C}} A$ means that $\mathcal{H}\otimes_{\mathbb{C}}A\cong A$ as a Hilbert A-module. We have $\mathbb{K}(A)=A$ and $\mathbb{K}(\mathcal{H}\otimes_{\mathbb{C}}A)\cong\mathbb{K}(\mathcal{H})\otimes A$. These are only isomorphic if $\mathcal{H}$ has dimension 1, and then A must act on $\mathcal{H}=\mathbb{C}$ by the augmentation character. So $\mathcal{E}$ is isomorphic to the correspondence associated to ε.

9. Polar decomposition

We are going to generalise the polar decomposition to operators on Hilbert modules. It is a useful tool to get unitary operators from non-unitary ones. It will be used to prove Kasparov's Stabilisation Theorem.

First let $V\colon\mathcal{H}_1\to\mathcal{H}_2$ be an operator between Hilbert spaces. Then $V=U|V|$, where U is a partial isometry and $|V|=(V^*V)^{1/2}$. This decomposition is natural; therefore, when we view V and U as operators on $\mathcal{H}_1\oplus\mathcal{H}_2$, then the partial isometry U belongs to the double commutant of V. But it need not belong to the multiplier algebra of the C*-algebra generated by V. Since $\mathbb{B}(B)\cong\mathcal{M}(B)$ for a C*-algebra B by Exercise 3.14, we cannot expect the polar decomposition to work for general adjointable operators between Hilbert modules.

Theorem 9.1. *Let B be a C*-algebra, let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert B-modules, and let $x\colon\mathcal{E}_1\to\mathcal{E}_2$ be an adjointable operator. If x and x^* have dense range, then there is a unique unitary operator $U\colon\mathcal{E}_1\to\mathcal{E}_2$ with $x=U|x|$. There is a partial isometry $U\colon\mathcal{E}_1\to\mathcal{E}_2$ with $x=U|x|$, $U(\mathcal{E}_1)=\overline{x(\mathcal{E}_1)}$, and $U^*(\mathcal{E}_2)=\overline{x^*(\mathcal{E}_2)}$ if and only if the ranges of x and x^* are complementable, and then U is unique.*

Proof. We copy the construction of the polar decomposition for Hilbert space operators. We want to define $U|x|(\xi):=x(\xi)$ for all $\xi\in\mathcal{E}$. We compute

$$\begin{aligned}\|x(\xi)\|^2&=\langle x(\xi)\,|\,x(\xi)\rangle=\langle\xi\,|\,x^*x(\xi)\rangle=\langle\xi\,|\,|x|^2(\xi)\rangle\\&=\langle|x|(\xi)\,|\,|x|(\xi)\rangle=\||x|(\xi)\|^2.\end{aligned}$$

Thus U is a well-defined, isometric map from $|x|(\mathcal{E}_1)$ to $x(\mathcal{E}_1)$. It extends to an isometry between the closures $U\colon\overline{x(\mathcal{E}_1)}\xrightarrow{\cong}\overline{|x|(\mathcal{E}_1)}$. Let U and U^* be zero on the orthogonal complements of $\overline{x(\mathcal{E}_1)}$ and $\overline{|x|(\mathcal{E}_1)}$, respectively. If the submodules $\overline{x(\mathcal{E}_1)}$ and $\overline{|x|(\mathcal{E}_1)}$ are complementable, then these extensions of U and U^* are defined everywhere; they are adjoints of each other, and $UU^*U=U$, $U^*UU^*=U^*$, that is, U is a partial isometry $\mathcal{E}_1 \to \mathcal{E}_2$. It is unitary if and only if both x and $|x|$ have dense range.

Next we check that $\overline{|x|(\mathcal{E}_1)} = \overline{x^*(\mathcal{E}_2)}$. Since $|x|$ is positive, it may be obtained by functional calculus from $|x|^\alpha$ for any $\alpha > 0$. Therefore, $\overline{|x|(\mathcal{E}_1)} = \overline{|x|^\alpha(\mathcal{E}_1)}$ for any $\alpha > 0$. Hence

$$\overline{|x|(\mathcal{E}_1)} = \overline{|x|^2(\mathcal{E}_1)} = \overline{x^*x(\mathcal{E}_1)} \subseteq \overline{x^*(\mathcal{E}_2)}.$$

We may write $x^* = (x^*x)^{1/4}y = |x|^{1/2}y$ for some $y \in \mathbb{B}(\mathcal{E}_2, \mathcal{E}_1)$ – this is possible in any C*-algebra, such as $\mathbb{B}(\mathcal{E}_1 \oplus \mathcal{E}_2)$. Hence $\overline{x^*(\mathcal{E}_2)} \subseteq \overline{|x|^{1/2}(\mathcal{E}_1)} = \overline{|x|(\mathcal{E}_1)}$.

If $U\colon \mathcal{E}_1 \to \mathcal{E}_2$ is a partial isometry, then UU^* and U^*U are projections onto $U(\mathcal{E}_1)$ and $U^*(\mathcal{E}_2)$, respectively, so these are complementable submodules. The condition $x = U|x|$ determines U on $\overline{|x|(\mathcal{E}_1)}$, and the condition $U^*(\mathcal{E}_2) = \overline{x^*(\mathcal{E}_2)}$ forces U to vanish on the complement of $\overline{x^*(\mathcal{E}_2)}$. Thus U is unique. □

Exercise 9.2. *Let $\mathcal{E}$ be a Hilbert module and let $\mathcal{F}$ be a countably generated Hilbert submodule. Show that there is a compact operator x on $\mathcal{E}$ with $\overline{x(\mathcal{E})} = \mathcal{F}$.*

Hence operators without polar decomposition exist whenever there are Hilbert submodules that are not complementable.

10. Kasparov's Stabilisation Theorem

Let B be a C*-algebra.

Definition 10.1. A Hilbert B-module $\mathcal{E}$ is *countably generated* if there is a countable subset $S \subseteq \mathcal{E}$ such that the linear span of $S \cdot B$ is dense in $\mathcal{E}$.

If the C*-algebra B is separable, then $\mathcal{E}$ is countably generated if and only if $\mathcal{E}$ is separable.

Theorem 10.2 (Kasparov's Stabilisation Theorem)**.** *If $\mathcal{E}$ is a countably generated Hilbert B-module, then $B^\infty \cong B^\infty \oplus \mathcal{E}$.*

This theorem justifies calling the Hilbert module B^∞ defined in (1.4) the *standard Hilbert B-module.*

By definition, B^∞ is the closure of $\sum_{n\in\mathbb{N}} B$ with respect to the norm from the inner product

$$\langle (a_n) \mid (b_n) \rangle = \sum_{n\in\mathbb{N}} a_n^* b_n.$$

Thus $(b_n) \in B^\infty$ if and only if $\sum_{n\in\mathbb{N}} b_n^* b_n$ is norm-convergent, that is, for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ with $\|\sum_N^M b_n^* b_n\| \le \varepsilon$ for all $M \ge N$. This condition is *strictly* weaker than $\sum \|b_n\|^2 < \infty$.

Let us compare the stabilisation theorem with Swan's Theorem for a compact space. It says that for any vector bundle E over X, there is a vector bundle $\tilde{E}$ such that $\tilde{E} \oplus E$ is trivial, that is, isomorphic to the vector bundle $X \times \mathbb{C}^n$ for some $n \in \mathbb{N}$. We cannot expect an isomorphism $E \oplus (X \times \mathbb{C}^n) \cong (X \times \mathbb{C}^n)$ for nontrivial X, of course. But we get

$$\begin{aligned} X \times \mathbb{C}^\infty &\cong (E \oplus \tilde{E}) \oplus (E \oplus \tilde{E}) + \cdots \\ &\cong E \oplus (\tilde{E} \oplus E) \oplus (\tilde{E} \oplus E) + \cdots \\ &\cong E \oplus X \times \mathbb{C}^\infty. \end{aligned}$$

Taking sections and completing to Hilbert modules gives Kasparov's Stabilisation Theorem for the Hilbert modules of sections of vector bundles. The above trick is known as an *Eilenberg swindle.* It is a variant of the Hilbert hotel. As the comparison with Swan's Theorem shows, it is crucial that B^∞ has infinite rank.

Proof of Theorem 10.2. The idea of the proof, which goes back to [20], is to construct an adjointable operator $x\colon B^\infty \to \mathcal{E} \oplus B^\infty$ such that x and x^* have dense range. Then the polar decomposition of x (Theorem 9.1) provides a unitary $B^\infty \to \mathcal{E} \oplus B^\infty$.

We pick a sequence (ξ_n) generating $\mathcal{E}$. By assumption, the map $B^\infty \to \mathcal{E}$, $(b_n) \mapsto \sum \xi_n b_n$, has dense range. Without further assumptions, this map may be unbounded. Even if it were adjointable, its adjoint would not have dense range because then we would get $\mathcal{E} \cong B^\infty$, which cannot be true in general. We must make our operator more complicated.

We may assume without loss of generality that $\|\xi_n\| \le 1$ for all $n \in \mathbb{N}$ and that each vector ξ_n is repeated infinitely often in the sequence because $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$. We define an operator $x\colon B^\infty \to \mathcal{E} \oplus B^\infty$ by sending $B^\infty \ni b_n\delta_n \mapsto (2^{-n}\xi_n b_n, 4^{-n} b_n \delta_n) \in \mathcal{E} \oplus B^\infty$. The projection of x to $\mathcal{E}$ is the operator

$$(2^{-n}|\xi_n\rangle)_{n\in\mathbb{N}}\colon B^\infty \to \mathcal{E}, \quad (b_n) \to \sum_{n\in\mathbb{N}} 2^{-n}\xi_n b_n,$$

which is compact and hence adjointable. The projection of x to B^∞ is the diagonal operator with diagonal entries 4^{-n}, which is also adjointable.

We claim that x has dense range. By construction, $2^{-n} \cdot (\xi_n b, 2^{-n} b\delta_n)$ is in the range of x for all $n \in \mathbb{N}$, $b \in B$. We may omit the factor 2^{-n}. Since each ξ_n is repeated infinitely often, we get $(\xi_n b, 2^{-m} b\delta_m)$ in the range for infinitely many m; letting $m \to \infty$, we get $(\xi_n b, 0)$ in the closed range for all $n \in \mathbb{N}$, $b \in B$. Then the closed range also contains $2^{-n} b\delta_n$ and hence $b\delta_n$. Since the sequence (ξ_n) generates $\mathcal{E}$, the closed range of x is all of $\mathcal{E} \oplus B^\infty$.

The adjoint of x maps $b\delta^n \mapsto 4^{-n} b\delta_n$. Hence x^* has dense range as well. Now the polar decomposition of X in Theorem 9.1 produces the required unitary. □

Definition 10.3. A C*-algebra B is *σ-unital* if it has a countable approximate unit.

Any C*-algebra contains an approximate unit, but this is a net in general. Being σ-unital ensures that we may take this net to be a sequence.

Lemma 10.4. *A* C**-algebra B is σ-unital if and only if it contains a* strictly positive element*: $a \in B_+$ with $\varphi(a) > 0$ for any state φ on B.*

Proof. Let (e_n) be an approximate unit. Then $\lim \varphi(e_n) = 1$ for any state φ. Hence $\sum 2^{-n} e_n$ is strictly positive.

Conversely, let $a \in B_+$ be strictly positive. Choose an increasing sequence (f_n) of functions $\mathbb{R}_+ \to [0,1]$ with $f_n(0) = 0$ for all $n \in \mathbb{N}$ and $f_n(t) \nearrow 1$ uniformly for $t \ge \varepsilon$ for all $\varepsilon > 0$. Then $(f_n(a))$ is an increasing sequence of positive elements in B. Since $\lim f_n(t)t = t$ uniformly for $t \in \mathbb{R}_+$, $\lim f_n(a) \cdot a = a$ in norm and hence $\lim f_n(a) \cdot b = b$ in norm if $b = ab'$ for some $b' \in B$. The set of elements b with norm-convergence $\lim f_n(a) \cdot b = b$ is norm-closed because $(f_n(a))$ is uniformly bounded. If the closure of $a \cdot B$ were not dense in B, then there would be a state vanishing on this right ideal, contradicting the strict positivity of a. Hence $a \cdot B$ is dense in B, and the convergence $\lim f_n(a) \cdot b = b$ works for all $b \in B$. □

Exercise 10.5. *Let $\mathcal{E}$ be a Hilbert module over a* C**-algebra B. A positive element of $\mathbb{K}(\mathcal{E})$ is strictly positive if and only if it has dense range as an operator on $\mathcal{E}$. The Hilbert module $\mathcal{E}$ is countably generated if and only if the* C**-algebra $\mathbb{K}(\mathcal{E})$ is σ-unital.*

Like B^∞, we may also define $\mathcal{E}^\infty$ for a Hilbert module $\mathcal{E}$: equip the algebraic direct sum $\sum_{n\in\mathbb{N}} \mathcal{E}$ with the inner product $\langle (\xi_n) \,|\, (\eta_n) \rangle_B := \sum_{n\in\mathbb{N}} \langle \xi_n \,|\, \eta_n \rangle_B$ and complete to a Hilbert B-module. Equivalently, $\mathcal{E}^\infty \cong \mathcal{E} \otimes \ell^2(\mathbb{N})$.

Theorem 10.6. *Let $\mathcal{E}$ be a full, countably generated Hilbert module over a σ-unital C^*-algebra B. Then $B^\infty \cong \mathcal{E}^\infty$.*

Proof. Kasparov's Stabilisation Theorem yields $B^\infty \cong B^\infty \oplus \mathcal{E}^\infty$ because $\mathcal{E}^\infty$ is still countably generated if $\mathcal{E}$ is. Recall that $\mathcal{E}^* := \mathbb{K}(\mathcal{E}, B)$ is a Hilbert module over $\mathbb{K}(\mathcal{E})$. We have $\mathbb{K}(\mathcal{E}^*) \cong B$ because $\mathcal{E}^*$ is full. Since $\mathcal{E}$ and $\mathcal{E}^*$ give a Morita–Rieffel equivalence between B and $\mathbb{K}(\mathcal{E})$, we have $\mathcal{E} \otimes_B \mathcal{E}^* \cong \mathbb{K}(\mathcal{E})$ and $\mathcal{E}^* \otimes_{\mathbb{K}(\mathcal{E})} \mathcal{E} \cong B$. Since B is assumed σ-unital, $\mathcal{E}^*$ is countably generated by Exercise 10.5. Hence so is $(\mathcal{E}^*)^\infty$. Thus Kasparov's Stabilisation Theorem gives

$$(\mathcal{E}^*)^\infty \oplus \mathbb{K}(\mathcal{E})^\infty \cong \mathbb{K}(\mathcal{E})^\infty.$$

Now we tensor this isomorphism with the correspondence $\mathcal{E}\colon \mathbb{K}(\mathcal{E}) \leftarrow B$ on the right. Since $\mathcal{E}^* \otimes_{\mathbb{K}(\mathcal{E})} \mathcal{E} \cong B$ and $\mathbb{K}(\mathcal{E}) \otimes_{\mathbb{K}(\mathcal{E})} \mathcal{E} \cong \mathcal{E}$, we get an isomorphism

$$B^\infty \oplus \mathcal{E}^\infty \cong \mathcal{E}^\infty.$$

Finally, we combine this with $B^\infty \cong B^\infty \oplus \mathcal{E}^\infty$. □

Definition 10.7. Two C^*-algebras A and B are *stably isomorphic* if $A \otimes \mathbb{K}(\ell^2(\mathbb{N})) \cong B \otimes \mathbb{K}(\ell^2(\mathbb{N}))$.

Theorem 10.8 (Brown–Green–Rieffel)**.** *Two σ-unital C^*-algebras are Morita–Rieffel equivalent if and only if they are stably isomorphic.*

Proof. We have $\mathbb{K}(\mathcal{E}^\infty) \cong \mathbb{K}(\mathcal{E}) \otimes \mathbb{K}(\ell^2\mathbb{N})$ for any Hilbert B-module $\mathcal{E}$ and, in particular, $\mathbb{K}(B^\infty) \cong \mathbb{K}(B) \otimes \mathbb{K}(\ell^2\mathbb{N})$. Hence a stable isomorphism between A and B is equivalent to an isomorphism $\mathbb{K}(A^\infty) \cong \mathbb{K}(B^\infty)$. Since A is Morita–Rieffel equivalent to $\mathbb{K}(A^\infty)$ and Morita–Rieffel equivalence is an equivalence relation, stable isomorphism implies Morita–Rieffel equivalence.

Conversely, if A and B are Morita–Rieffel equivalent, then $A \cong \mathbb{K}(\mathcal{E})$ for a full Hilbert B-module $\mathcal{E}$. Theorem 10.6 shows $\mathcal{E}^\infty \cong B^\infty$ as Hilbert B-modules. Hence

$$A \otimes \mathbb{K}(\ell^2\mathbb{N}) \cong \mathbb{K}(\mathcal{E}^\infty) \cong \mathbb{K}(B^\infty) \cong B \otimes \mathbb{K}(\ell^2\mathbb{N}). \qquad \square$$

A Morita–Rieffel equivalence bimodule $\mathcal{E}$ between A and B induces a *-isomorphism

$$A \otimes \mathbb{K} \cong \mathbb{K}(\mathcal{E}^\infty) \cong \mathbb{K}(B^\infty) \cong B \otimes \mathbb{K},$$

where $\mathbb{K} := \mathbb{K}(\ell^2\mathbb{N})$. Here the isomorphisms $A \otimes \mathbb{K} \cong \mathbb{K}(\mathcal{E}^\infty)$ and $B \otimes \mathbb{K} \cong \mathbb{K}(B^\infty)$ are canonical. The isomorphism $\mathbb{K}(\mathcal{E}^\infty) \cong \mathbb{K}(B^\infty)$ depends on the choice of the Hilbert module isomorphism $\mathcal{E}^\infty \cong B^\infty$. Two different isomorphisms $\mathcal{E}^\infty \cong B^\infty$ differ by a unitary $u \in \mathbb{B}(B^\infty) \cong \mathcal{M}(B \otimes \mathbb{K})$. Hence the *-isomorphism $A \otimes \mathbb{K} \to B \otimes \mathbb{K}$ is unique up to composition with an inner automorphism Ad_u.

As an application, consider a Morita–Rieffel equivalence $\mathcal{E}$ from a C^*-algebra A to itself. Then $\mathcal{E}$ induces a *-automorphism α of $A \otimes \mathbb{K}$. Since α is unique up to inner automorphisms, the resulting covariance algebra $(A \otimes \mathbb{K}) \rtimes_\alpha \mathbb{Z}$ is unique up to isomorphism. This suggests that there should be a covariance algebra $A \rtimes_{\mathcal{E}} \mathbb{Z}$ with $(A \otimes \mathbb{K}) \rtimes_\alpha \mathbb{Z} \cong (A \rtimes_{\mathcal{E}} \mathbb{Z}) \otimes \mathbb{K}$.

How much can we say about non-invertible correspondences?

Let $\mathcal{E}\colon A \leftarrow B$ be a correspondence. Let $I := \langle \mathcal{E} \,|\, \mathcal{E} \rangle_B \triangleleft B$ be the ideal such that $\mathcal{E}$ is a full Hilbert I-module. Even if we assume A and B σ-unital, $\mathcal{E}$ need not be countably generated. For instance, for $A = B = \mathbb{C}$, arbitrarily large Hilbert spaces give a correspondence $A \leftarrow B$. To get anywhere, we better assume $\mathcal{E}$ countably generated as well. Another problem is that ideals in σ-unital C^*-algebras need not be σ-unital because I^+ is unital and hence σ-unital for any C^*-algebra I. To apply Theorem 10.6, we must assume that $\mathcal{E}$ is countably generated as a Hilbert B-module and that I is σ-unital; since $I \cong \mathbb{K}(\mathcal{E}^*)$, this is equivalent to $\mathcal{E}^*$ being countably generated as a Hilbert $\mathbb{K}(\mathcal{E})$-module. Then we get an isomorphism of

Hilbert modules $\mathcal{E}^\infty \cong I^\infty$. The left action $A \to \mathbb{B}(B)$ gives a nondegenerate *-homomorphism

$$A \otimes \mathbb{K} \to \mathbb{B}(\mathcal{E}^\infty) \cong \mathbb{B}(I^\infty) \cong \mathcal{M}(\mathbb{K}(I^\infty)) \cong \mathcal{M}(I \otimes \mathbb{K}).$$

Thus a correspondence $\mathcal{E}\colon A \leftarrow B$ with both $\mathcal{E}$ and $\mathcal{E}^*$ countably generated gives a morphism

$$A \otimes \mathbb{K} \to I \otimes \mathbb{K} \triangleleft B \otimes \mathbb{K}$$

for the ideal $I = \langle \mathcal{E} \mid \mathcal{E}\rangle_B \triangleleft B$. This morphism is unique up to composition with Ad_u for $u \in \mathcal{U}(I \otimes \mathbb{K})$.

Exercise 10.9. *If $I \triangleleft B$, there is a strictly continuous, unital *-homomorphism $\mathcal{M}(B) \to \mathcal{M}(I)$. When is it injective? Find an example where this map is injective but not surjective.*

We can use the above description of correspondences to find a *strict* 2-category equivalent to the correspondence bicategory at least for σ-unital C*-algebras. Its arrows $A \leftarrow B$ are pairs (I, φ) with $I \triangleleft B$ and a morphism $\varphi\colon A \otimes \mathbb{K} \to I \otimes \mathbb{K}$. The 2-arrows $(I, \varphi) \Rightarrow (J, \psi)$ are unitary multipliers of $I \otimes \mathbb{K}$ that intertwine φ and ψ if $I = J$; there are no 2-arrows if $I \neq J$. The composition of these arrows is done to match the composition of correspondences. But now the composition is really associative, not just up to 2-arrows, and the identity arrow $(B, \mathrm{id}_{B\otimes\mathbb{K}})$ on B is really a strict unit arrow on B. We will not use this strictification of the correspondence bicategory because it seems artificial.

Definition 10.10. A C*-algebra A is C*-*stable* if $A \cong \mathbb{K}(\ell^2\mathbb{N}) \otimes A$.

Since $\mathbb{K}(\ell^2\mathbb{N}) \otimes \mathbb{K}(\ell^2\mathbb{N}) \cong \mathbb{K}(\ell^2\mathbb{N})$, C*-algebras of the form $\mathbb{K}(\ell^2\mathbb{N}) \otimes A$ are always C*-stable.

The Brown–Green–Rieffel Theorem 10.8 implies that two C*-stable, σ-unital C*-algebras are Morita–Rieffel equivalent if and only if they are isomorphic.

Exercise 10.11. *If A is* C*-*stable and $\mathcal{E}\colon A \leftarrow B$ is a correspondence, then $\mathcal{E} \cong \mathcal{E}^\infty$ as a correspondence. If $\mathcal{E}$ is full and countably generated and B is also* C*-*stable, then $\mathcal{E} \cong B$ as a Hilbert module, so that $\mathcal{E} \cong B_f$ for some morphism $f\colon A \to B$.*

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