Noncommutative Geometry IV: Differential Geometry

1. Coordinates and manifolds

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How to work with this course?

- The 24 sections in the notes correspond to the lectures.
- Read a section in the notes before the corresponding class.
- Note questions that you want to discuss with me.
- If possible, write these questions in the Stud.IP forum.
  Then I can prepare some notes about these topics.

Noncommutative geometry

- Noncommutative geometry considers noncommutative algebras as if they would describe geometric objects.
- This course will discuss examples of noncommutative algebras and some properties and structures of algebras that are analogous to familiar geometric structures.
- Irreducible representations of algebras are related to points of a classical space.
- Derivations are related to tangent vectors and vector fields on manifolds.
- Hochschild cohomology is a sequence of invariants. Its first entries are related to derivations and to deformations.
- Periodic cyclic homology is an analogue of de Rham cohomology of manifolds.

Algebraic geometry

- Before we consider noncommutative algebras as geometric objects, we explain in the first three lectures how to turn ordinary geometric objects into commutative algebras.
- We shall do this first with smooth manifolds; then, more briefly, with algebraic varieties.
- The first lecture on coordinates and manifolds recalls the concept of smooth manifold.
- The second lecture explains how to recover the manifold from its algebra of smooth functions.
- The third lecture briefly explains the analogous theory for algebraic varieties.

Coordinates and manifolds

- A smooth manifold is a space with nice local coordinates.
- The precise definition is the central concept in this lecture.
- Before that, we recall why coordinates are important.
- The definition of a smooth manifold also contains some technical extra conditions besides the local coordinates. A good justification for them is that any smooth manifold embeds into $\mathbb{R}^n$.

Topological manifolds

Definition

A *d-dimensional topological manifold* is a paracompact, Hausdorff topological space for which each point has an open neighbourhood homeomorphic to $\mathbb{R}^d$.
Such a homeomorphism is a chart.
Its domain is a chart neighbourhood.
Let $\varphi\colon U \to \mathbb{R}^d$ and $\psi\colon V \to \mathbb{R}^d$ be charts on overlapping chart neighbourhoods. The change of coordinate map

$$\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$$

transforms coordinates between the two local coordinate systems on the overlap $U \cap V$.

Smooth manifolds

Definition

An atlas on a d-dimensional topological manifold M is a family of charts $\{\varphi\colon U_\varphi \to \mathbb{R}^d\}$ such that the domains U_φ cover M and all change of coordinate maps are smooth. A smooth manifold is a topological manifold together with an atlas.

Definition

A function $f\colon M \to \mathbb{R}^k$ is smooth if $f \circ \varphi^{-1}\colon \mathbb{R}^d \to \mathbb{R}^k$ is smooth for all coordinate charts in the atlas.

Definition

A map $f\colon X \to Y$ is called smooth if it is continuous and if,
for all $x \in X$ and smooth charts $\varphi\colon U_x \to \mathbb{R}^{d_X}$, $\psi\colon U_{f(x)} \to \mathbb{R}^{d_Y}$
on neighbourhoods U_x and $U_{f(x)}$ of x and $f(x)$,
the map $\psi \circ f \circ \varphi^{-1}$ from $\varphi\big(f^{-1}(U_{f(x)})\big) \subseteq \mathbb{R}^{d_X}$ to $\mathbb{R}^{d_Y}$ is smooth.

The Embedding Theorem

Definition

An embedding $X \hookrightarrow Y$ for two smooth manifolds X and Y is an injective, smooth map $f\colon X \to Y$
that is a homeomorphism onto its image
and whose first derivatives – computed in local coordinates – are injective at each point.

Theorem (Embedding Theorem)

Let X be a smooth d-dimensional manifold.
There is a proper embedding $X \to \mathbb{R}^N$ for some $N \in \mathbb{N}$.
Even more, we can choose $N = 2d + 1$.

Remark

Any closed subset of $\mathbb{R}^N$ is paracompact and Hausdorff.
So these two extra assumptions in the definition of a manifold are necessary for the embedding theorem.