Noncommutative Geometry IV: Differential Geometry

2. Recovering a manifold from its algebra of smooth functions

R. Meyer

Mathematisches Institut
Universität Göttingen

Summer Term 2020

Algebraic geometry

- Before we consider noncommutative algebras as geometric objects, we explain in the first three lectures how to turn ordinary geometric objects into commutative algebras.
- We shall do this first with smooth manifolds; then, more briefly, with algebraic varieties.
- The first lecture on coordinates and manifolds recalls the concept of smooth manifold.
- The second lecture explains how to recover the manifold from its algebra of smooth functions.
- The third lecture briefly explains the analogous theory for algebraic varieties.

Recovering a manifold from its algebra of smooth functions

- We recall the concept of a K-algebra over a field K.
- Let X be a smooth manifold and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
  The smooth functions $X \to \mathbb{K}$ form a $\mathbb{K}$-algebra $\mathrm{C}^\infty(X, \mathbb{K})$.
- A character on an K-algebra A is a non-zero algebra homomorphism $A \to K$.
- The map sending $x \in X$ to the evaluation homomorphism $\mathrm{ev}_x \colon \mathrm{C}^\infty(X, \mathbb{K}) \to \mathbb{K}$ is a bijection from X onto the set of characters on $\mathrm{C}^\infty(X, \mathbb{K})$.
- The set of characters on an algebra carries a canonical topology, called the Zariski topology.
- The Zariski topology on the set of characters on $\mathrm{C}^\infty(X, \mathbb{K})$ is the usual topology on X.
- A smooth map $f \colon X \to Y$ induces an algebra homomorphism $\mathrm{C}^\infty(Y, \mathbb{K}) \to \mathrm{C}^\infty(X, \mathbb{K})$, and any algebra homomorpism is of this form for a unique smooth map.
- The isomorphism class of the algebra $\mathrm{C}^\infty(X, \mathbb{K})$ determines X up to diffeomorphism.

Algebras over fields

Definition

Let K be a field. A K-algebra is a K-vector space A with a map

$$m\colon A \times A \to A, \qquad (x, y) \mapsto x \cdot y,$$

called multiplication, which is bilinear and associative:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad \text{for all } a,\ b,\ c \in A. \tag{1}$$

It is commutative if

$$a \cdot b = b \cdot a \qquad \text{for all } a,\ b \in A. \tag{2}$$

A unit element is an element $1_A \in A$ with

$$1_A \cdot a = a = a \cdot 1_A \qquad \text{for all } a \in A. \tag{3}$$

An algebra with a unit element is called unital.

Algebras of smooth functions

Example

Let X be a smooth manifold and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathrm{C}^\infty(X, \mathbb{K})$ be the set of smooth functions $X \to \mathbb{K}$.
This is a commutative, unital $\mathbb{K}$-algebra for the pointwise addition, scalar multiplication, and multiplication:

$$
\begin{aligned}
(f_1 + f_2)(x) &:= f_1(x) + f_2(x), \\
(\lambda \cdot f)(x) &:= \lambda \cdot f(x), \\
(f_1 \cdot f_2)(x) &:= f_1(x) \cdot f_2(x)
\end{aligned}
$$

for $f_1, f_2, f \in \mathrm{C}^\infty(X, \mathbb{K})$, $\lambda \in \mathbb{K}$.

We write $\mathrm{C}^\infty(X)$ if $\mathbb{K}$ is unimportant.

Characters

Definition

Let K be a field and let A and B be K-algebras.
An algebra homomorphism from A to B is a map $f\colon A \to B$ that is K-linear and satisfies $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in A$.
If A and B are unital, we call f unital if $f(1_A) = 1_B$.

Definition

Let K be a field. A (K-valued) character on a K-algebra A is a non-zero algebra homomorphism $A \to K$.
Let $\hat{A}$ be the set of characters on A.

Theorem

Let X be a smooth manifold and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For each $x \in X$,

$$\mathrm{ev}_x\colon \mathrm{C}^\infty(X, \mathbb{K}) \to \mathbb{K}, \qquad f \mapsto f(x),$$

is a character. Any character is of this form. This yields a canonical bijection between X and the set of characters on $\mathrm{C}^\infty(X, \mathbb{K})$.

The Zariski topology on the space of characters

Definition

Let A be a commutative K-algebra. Let S be a subset of $\hat{A}$. Its Zariski closure $\overline{S}$ is the set of all $\chi \in \hat{A}$ with $\chi(a) = 0$ for all $a \in A$ with $\omega(a) = 0$ for all $\omega \in S$.
Equivalently, $\overline{S}$ is the largest subset such that $\omega(a) = 0$ for all $\omega \in S$ implies $\chi(a) = 0$ for all $\chi \in \overline{S}$.

Lemma

The Zariski closure satisfies the Kuratowski closure axioms and thus defines a topology on $\hat{A}$.

Lemma

The canonical bijection between X and the character space of $C^\infty(X)$ becomes a homeomorphism for the Zariski topology.

Proposition

A map $f\colon X \to Y$ *is smooth if and only if* $h \mapsto h \circ f$ *defines a unital algebra homomorphism* $f^*\colon \mathrm{C}^\infty(Y) \to \mathrm{C}^\infty(X)$*.*
This defines a bijection from smooth maps $X \to Y$ *to unital algebra homomorphisms* $\mathrm{C}^\infty(Y) \to \mathrm{C}^\infty(X)$*.*

In the language of category theory, $X \mapsto \mathrm{C}^\infty(X, \mathbb{K})$ is a fully faithful functor from the category of smooth manifolds and smooth maps to the category of unital $\mathbb{K}$-algebras and unital algebra homomorphisms.

Corollary

If $\mathrm{C}^\infty(X)$ *and* $\mathrm{C}^\infty(Y)$ *are isomorphic algebras, then* X *and* Y *are diffeomorphic.*

Topology on $C^\infty(X)$

Definition

A sequence $(f_n)_{n\in\mathbb{N}}$ in $C^\infty(X)$ converges to $f \in C^\infty(X)$ if and only if for each chart $\varphi\colon U \to \mathbb{R}^d$, each compact subset $L \subseteq \varphi(U)$, and each multi-index α, the sequence of functions $\partial^\alpha(f_n \circ \varphi^{-1})|_L\colon L \to \mathbb{K}$ converges uniformly towards $\partial^\alpha(f \circ \varphi^{-1})|_L$.

Proposition

Any continuous character on $C^\infty(X)$ *is* ev_x *for some* $x \in X$.

The proof is done by contradiction. Let $\chi\colon C^\infty(X, \mathbb{K}) \to \mathbb{K}$ be a character with $\chi \neq \mathrm{ev}_x$ for all $x \in X$.
We are going to prove that $\chi(1) = 0$, which is impossible.

Proof continued

- By assumption, there is $f \in C^\infty(X)$ with $\chi(f) \neq f(x)$.
- Since $\chi(1) = 1$, we may subtract $\chi(f) \cdot 1$ from f to get $f_x \in C^\infty(X)$ with $\chi(f_x) = 0$ and $f_x(x) \neq 0$.
- Then $\chi(f_x \cdot \overline{f_x}) = 0$ as well.
  And $|f_x|^2 = f_x \cdot \overline{f_x}$ is positive in a neighbourhood U_x of x.
- Let L be a compact subset of X. Then $L \subseteq \bigcup_{i=1}^n U_{x_i}$ for finitely many of these open subsets.
- Then $\sum_{i=1}^n |f_{x_i}|^2$ is positive on L and in the kernel of χ.
- Multiplying with a suitable function, we get a smooth function g_L with $g_L|_L = 1$ and $\chi(g_L) = 0$.
- These functions g_L converge to 1 if $L \to X$.
- Since χ is continuous, it follows that $\chi(1) = 0$.
- This is impossible. So there must be $x \in X$ with $\chi = \mathrm{ev}_x$.