Noncommutative Geometry IV: Differential Geometry

3. Algebraic varieties

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Algebraic geometry

- Before we consider noncommutative algebras as geometric objects, we explain in the first three lectures how to turn ordinary geometric objects into commutative algebras.
- We shall do this first with smooth manifolds; then, more briefly, with algebraic varieties.
- The first lecture on coordinates and manifolds recalls the concept of smooth manifold.
- The second lecture explains how to recover the manifold from its algebra of smooth functions.
- The third lecture briefly explains the analogous theory for algebraic varieties.

Algebraic varieties

- We define affine algebraic varieties.
- We relate them to radical ideals in polynomial algebras.
- We sketch how affine algebraic varieties correspond to radical, finitely generated, commutative algebras.
- We generalise to affine algebraic varieties over $\mathbb{R}$ and discuss their real and complex points.
- We modify the concept of character to allow characters into field extensions of the ground field.
- We find a bijection between characters and maximal ideals.
- We show that the new characters on $C^\infty(X)$ are still just point evaluations if X is compact.
- If X is non-compact, this is true for continuous characters on $C^\infty(X)$.

Affine algebraic varieties

Definition

An affine complex algebraic variety is a subset of $\mathbb{C}^n$
that is defined by algebraic equations.
That is, it is the solution set of a set of polynomial equations.

An affine complex algebraic variety is given by its vanishing ideal

$$I_V := \{p \in \mathbb{C}[x_1, \ldots, x_n] : p|_V = 0\}$$

in $\mathbb{C}[x_1, \ldots, x_n]$.

Theorem (Hilbert's Nullstellensatz)

An ideal in $\mathbb{C}[x_1, \ldots, x_n]$ is of the form I_V for a subset $V \subseteq \mathbb{C}^n$ if and only if it is radical:
if $p^n \in I_V$ for some $n \in \mathbb{N}$, then $p \in I_V$.

Regular functions on affine algebraic varieties

Definition

The algebra of polynomial (usually called regular) functions on V is

$$\mathrm{Pol}(V) := \mathbb{C}[x_1, \ldots, x_n]/I_V.$$

It describes V independently of an embedding in $\mathbb{C}^n$.

Theorem

A $\mathbb{C}$-algebra is of the form $\mathrm{Pol}(V)$ for an affine complex algebraic variety V if and only if it is commutative and finitely generated and its radical vanishes.

Example

The algebra $\mathbb{C}[x]/(x^2)$ is commutative and finitely generated, but not radical.

Affine algebraic varieties over the reals

Definition

Let K be any field. An affine algebraic variety in K^n is a radical ideal I in $K[x_1, \ldots, x_n]$. The algebra of regular functions on the variety is the quotient algebra $\mathrm{Pol}(V) := K[x_1, \ldots, x_n]/I$.

Example

Let $I = (x^2 + 1) \subseteq \mathbb{R}[x]$. The algebra of regular functions is $\mathbb{R}[x]/I \cong \mathbb{C}$. There is no character $\mathbb{R}[x]/I \to \mathbb{R}$ because $x^2 + 1 = 0$ has no real solutions. There are two characters $\mathbb{C} \cong \mathbb{R}[x]/I \to \mathbb{C}$: the identity map and complex conjugation.
The real algebraic variety defined by the equation $x^2 + 1 = 0$ has no real points and two complex points i and $-$i.

Characters and maximal ideals

Definition

Let K be a field and let A be a K-algebra.
A character on A is a surjective unital homomorphism $A \to L$ for some field extension L of K.
Two characters $\varphi\colon A \to L$ and $\varphi'\colon A \to L'$ are considered equivalent if there is an isomorphism $\lambda\colon L \to L'$ with $\lambda \circ \varphi = \varphi'$ and $\lambda|_K = \mathrm{Id}_K$.

Definition

A maximal ideal in a K-algebra A is a proper ideal $I \subsetneq A$ for which there is no ideal J with with $I \subsetneq J \subsetneq A$.

Proposition

Let A be a commutative, unital K-algebra.
Mapping a character χ to its kernel $\ker \chi$ defines a bijection between the set of maximal ideals in A and the set of equivalence classes of characters on A.

Maximal ideals in the algebra of smooth functions

Proposition

Let X be a smooth compact manifold. Any ideal in $\mathrm{C}^\infty(X)$ is contained in $I_x := \{f \in \mathrm{C}^\infty(X) : f(x) = 0\}$ for some $x \in X$. If X is not compact, then $\mathrm{C}^\infty(X)$ has maximal ideals that are not of this form. These contain $\mathrm{C}_c^\infty(X)$ and are dense in $\mathrm{C}^\infty(X)$. And the quotient field has uncountable dimension.

Algebraic varieties that are not affine

Theorem (Liouville's Theorem)

The only holomorphic functions on a compact complex manifold are the constant ones.

- Thus we cannot describe complex manifolds through its algebra of global holomorphic functions.
- We cannot describe a compact algebraic variety through an algebra of regular functions.
- Projective varieties are described through graded algebras.
- We may use sheaf theory or differential graded algebras.
  Differential graded algebras have also been used in noncommutative differential geometry.
  They are not treated much in this course.