Noncommutative Geometry IV: Differential Geometry

4. Representations and simple modules

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Summer Term 2020

Representations and simple modules

Question

What could replace the points for a noncommutative space?

- Maximal ideals make sense for them as well.
- We may also replace characters by simple modules or, equivalently, irreducible representations.
- The kernels of irreducible representations are primitive ideals.
- These are a bit more general than maximal ideals.
- In this lecture, we will introduce basic notions of representation theory and apply them to simple examples.

Modules generalise linear algebra

- As two important examples in this lecture, we will study modules over the algebra of polynomials $\mathbb{C}[x]$ and over the algebra of upper triangular 2×2-matrices.
- These turn out to be equivalent to linear maps $V \to V$ and to linear maps $V_0 \to V_1$, respectively, where V, V_0, V_1 are $\mathbb{C}$-vector spaces.
- Thus the study of modules over noncommutative algebras generalises linear algebra.
- This is an important motivation to study noncommutative algebras.

Definition

Let K be a field and let V be a K-vector space.
Let $\mathrm{End}(V)$ be the space of K-linear maps $V \to V$.
This is a K-algebra with respect to composition of maps as multiplication and pointwise addition and scalar multiplication.
A representation of a K-algebra A on V is an algebra homomorphism $f\colon A \to \mathrm{End}(V)$.
The pair (V, f) is also called a (left) A-module.
A representation or module is called faithful if f is injective.

Example

Let A be a K-algebra.
The regular representation of A on itself is the representation $\lambda\colon A \to \mathrm{End}(A)$ defined by $\lambda_a(b) := a \cdot b$.
It is faithful if A is unital.

Polynomials in one variable

Example

Let $A = \mathbb{C}[x]$. A representation $f : A \to \operatorname{End}(V)$ of A on V is determined by $f(x)$, which may be any K-linear map $V \to V$.
Since A has infinite dimension, a linear map $V \to V$ can only give a faithful A-module if V is infinite-dimensional.
If a module is not faithful, then there is a non-zero polynomial $p \in \mathbb{C}[x]$ with $f(p) = 0$. The normalised polynomial of smallest degree generates the kernel of f.
It is the minimal polynomial of $f(x)$.
A vector $v \in V$ is an eigenvector if and only if the subspace $\mathbb{C} \cdot v$ is an A-submodule of V.

Upper triangular 2×2-matrices

Theorem

Let A be the unital K-algebra of upper triangular matrices in $\mathbb{M}_2 K$. An A-module is a vector space with linear maps P, S that satisfy $P^2 = P$, $P \cdot S = 0$ and $S \cdot P = S$.
The category of unital A-modules with A-linear maps as morphisms is equivalent to the category whose objects are the K-linear maps between K-vector spaces and whose morphisms from T to T' are the commuting diagrams

$$\begin{array}{ccc} V_0 & \xrightarrow{T} & V_1 \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} \\ V_0' & \xrightarrow{T'} & V_1', \end{array}$$

that is, pairs of maps (φ_0, φ_1) with $T' \circ \varphi_0 = \varphi_1 \circ T$.

Submodules and simple modules

Definition

Let $(V, f\colon A \to \mathrm{End}(V))$ be an A-module.
An A-submodule is a vector subspace W of V for which $f(a)(w) \in W$ for all $a \in A$, $w \in W$; then f restricts to a map $A \to \mathrm{End}(W)$ that turns W into an A-module.
We call (V, f) with $f \neq 0$ a simple A-module or an irreducible representation of A if the only A-submodules are $\{0\}$ and V.

Definition

An algebra is called (left) primitive if it has a faithful simple module. An ideal $I \triangleleft A$ is called primitive if A/I is primitive or, equivalently, there is a simple module (V, f) with $\ker f = I$.

Definition

Let A be a K-algebra. The set of primitive ideals of A is called the primitive ideal space and denoted by $\mathrm{Prim}(A)$. The set of isomorphism classes of simple A-modules is denoted by $\hat{A}$.

Results on simple modules

Proposition

Any unital K-algebra A has a simple module.
Simple unital K-algebras are primitive.
Maximal ideals in unital algebras are primitive ideals.

Corollary

Let A be a commutative unital K-algebra.
The map $\hat{A} \to \mathrm{Prim}(A)$ is bijective.
And $\mathrm{Prim}(A)$ is the set of maximal ideals in A.
These are in bijection with characters $A \to L$ for fields $L \supseteq K$.