# Noncommutative Geometry IV: Differential Geometry

## 5. Endomorphism algebras and finite-dimensional algebras

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# Endomorphism algebras and finite-dimensional algebras

We will study modules and ideals for two classes of algebras:

1. Algebras of endomorphisms
2. Finite-dimensional algebras

# Algebras of endomorphisms

- The algebra $\mathrm{End}(V)$ contains an important ideal $\mathrm{End}_{\mathrm{f}}(V)$, spanned by rank-1-operators.
- If $V$ has a countable basis, then $\mathrm{End}_{\mathrm{f}}(V)$ is the only non-trivial ideal in $\mathrm{End}(V)$.
- The algebra $\mathrm{End}_{\mathrm{f}}(V)$ has only one irreducible representation: its representation on $V$.
- Any nondegenerate representation of $\mathrm{End}_{\mathrm{f}}(V)$ is a direct sum of copies of the representation on $V$.
- The category of nondegenerate representation of $\mathrm{End}_{\mathrm{f}}(V)$ is equivalent to the category of vector spaces.

# Finite-Rank Operators

Let $V^* := \mathrm{Hom}_K(V, K)$ be the dual vector space.
Given $\widehat{v} \in V^*$ and $w \in W$, we define $K$-linear maps

$$\begin{aligned} |w\rangle \colon K \to W, &\qquad \kappa \mapsto w \cdot \kappa, \\ \langle\widehat{v}| \colon V \to K, &\qquad v \mapsto \widehat{v}(v), \\ |w\rangle\langle\widehat{v}| \colon V \to W, &\qquad v \mapsto |w\rangle\big(\langle\widehat{v}|(v)\big) = w \cdot \widehat{v}(v). \end{aligned} \tag{1}$$

Any operator $V \to W$ of rank 1 is of the form $|w\rangle\langle\widehat{v}|$ for some $w \in W \setminus \{0\}$ and some $\widehat{v} \in V^* \setminus \{0\}$.
And $T \circ |w\rangle\langle\widehat{v}| \circ S = |T(w)\rangle\langle S^*\widehat{v}|$.

# Ideals in $\mathrm{End}(V)$

## Proposition

*Any non-zero ideal in* $\mathrm{End}(V)$ *contains* $\mathrm{End}_{\mathrm{f}}(V)$.
*If* $V$ *has countably infinite dimension,*
*then* $\{0\}$, $\mathrm{End}_{\mathrm{f}}(V)$, *and* $\mathrm{End}(V)$ *are the only ideals in* $\mathrm{End}(V)$.

## Proof.

- Let $I$ be a non-zero ideal, $T \in I \setminus \{0\}$.
- Then there is $v \in V$ with $T(v) \neq 0$.
- Then there is $\widehat{v} \in V^*$ with $\widehat{v}(T(v)) = 1$.
- If $w \in V$, $\widehat{w} \in V^*$, then $|w\rangle\langle\widehat{v}|T|v\rangle\langle\widehat{w}| = |w\rangle\langle\widehat{w}| \in I$.
- Then $\mathrm{End}_{\mathrm{f}}(V) \subseteq I$.
- Assume also $I \neq \mathrm{End}_{\mathrm{f}}(V)$. Then $I$ contains $T$ of infinite rank.
- Choose countable bases for $V$ and the image of $T$ to find $a, b \in \mathrm{End}(V)$ with $aTb = 1$.

□

# Nondegenerate modules

### Definition

Let $V$ be a vector space and $A$ an algebra.
A representation $f: A \to \mathrm{End}(V)$ is non-degenerate if elements of the form $f(a)(v)$ for $a \in A$, $v \in V$ span $V$ or, briefly, $A \cdot V = V$.

### Lemma

*Let $A$ be unital with unit $1_A$. An $A$-module is non-degenerate if and only if $1_A \cdot v = v$ for all $v \in V$.*

### Example

The obvious representation $\mathrm{End}_{\mathrm{f}}(V) \to \mathrm{End}(V)$ is non-degenerate.

### Theorem

*The category of non-degenerate representations of $\mathrm{End}_{\mathrm{f}}(V)$ is equivalent to the category of vector spaces.*

# Finite-Dimensional algebras

- Let $A$ be a finite-dimensional $\mathbb{C}$-algebra.
- There is a maximal ideal $I \subseteq A$ that is nilpotent – $I^n = A$.
- This ideal is also the intersection of the kernels of all irreducible representations of $A$.
- The quotient $A/I$ is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$.
- If $A$ is a finite-dimensional C$^*$-algebra, then it contains no nilpotent ideals. Then $A$ is isomorphic to a sum of matrix algebras.

# Some classical theorems

### Theorem (Wedderburn)

*A finite-dimensional $K$-algebra for a field $K$ is simple if and only if it is isomorphic to $\mathbb{M}_n D$ for some $n \in \mathbb{N}_{\geq 1}$ and some finite-dimensional division algebra $D$ over $K$.*

### Theorem (Frobenius)

*Any finite-dimensional division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or to the algebra $\mathbb{H}$ of quaternions.*

# The nilradical

### Definition

Let $A$ be an algebra. An ideal $I \triangleleft A$ is called nilpotent if there is $n \in \mathbb{N}$ such that $i_1 \cdots i_n = 0$ for all $i_1, \ldots, i_n \in I$.

### Lemma

*Let $I_1, I_2 \triangleleft A$ be nilpotent ideals. Then*

$$I_1 + I_2 := \{i_1 + i_2 : i_1 \in I_1, i_2 \in I_2\}$$

*is a nilpotent ideal as well. Therefore, any algebra contains a maximal nilpotent ideal – its nilradical.*

# The radical

**Lemma**

*Let $A$ be an algebra.*
*The nilradical is contained in all primitive ideals of $A$.*

The intersection of all primitive ideals is also called radical.

**Proof.**

- Let $I \triangleleft A$ be a nilpotent ideal and let $f \colon A \to \mathrm{End}(V)$ be an irreducible representation of $A$. We prove by contradiction that $f|_I = 0$.
- Otherwise, $I \cdot V = V$ by irreducibility.
- Then $I^k \cdot V = V$ for all $k \in \mathbb{N}$ by induction.
- This is impossible because $I^k = 0$ for some $k \in \mathbb{N}$. □

# Finite-Dimensional semi-simple algebras

## Definition

A finite-dimensional algebra is semi-simple if its nilradical vanishes or, equivalently, if zero is its only nilpotent ideal.

## Theorem

*Any semi-simple finite-dimensional algebra is isomorphic to a finite direct product of simple algebras, that is, to a finite direct product of matrix algebras over division algebras.*

Let $A$ be any finite-dimensional algebra.
Let $\operatorname{rad} A \triangleleft A$ be its nilradical. Then $A/\operatorname{rad} A$ is semi-simple.
Hence it is isomorphic to a finite direct product of matrix algebras over division algebras:

$$A/\operatorname{rad} A \cong \bigoplus_{j=1}^{n} \mathbb{M}_{m_j} D_j$$

for some $n \in \mathbb{N}_{\geq 0}$, $m_j \in \mathbb{N}_{\geq 1}$, and division algebras $D_j$ over $K$.
The radical is equal to the nilradical.

# Pointlike invariants for finite-dimensional algebras

**Theorem**

*The representations $A/\operatorname{rad} A \twoheadrightarrow \mathbb{M}_{m_j} D_j$ for $j = 1, \ldots, n$ form a set of representatives for $\widehat{A}$.*

*Both $\widehat{A}$ and $\operatorname{Prim}(A)$ consist of exactly $n$ points.*

*Any primitive ideal in $A$ is maximal.*

# Finite-Dimensional C*-algebras

### Theorem

*Let $A$ be a finite-dimensional unital algebra and let $\langle \textvisiblespace \mid \textvisiblespace \rangle$ be an inner product on $A$ with the property that for each $a \in A$, there is $a^* \in A$ with $\langle ax \mid y \rangle = \langle x \mid a^* y \rangle$. Then $A$ is semi-simple.*

### Proposition

*Let $A \subseteq \mathbb{M}_n\mathbb{C}$ be a subalgebra with $x^* \in A$ for all $x \in A$; here the adjoint of a matrix is defined by $(x_{ij})^* := (\overline{x_{ji}})$.*
*Then $\operatorname{rad} A = \{0\}$. So $A$ is a direct sum of matrix algebras.*