Noncommutative Geometry IV: Differential Geometry

6. Group algebras

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Group algebras

- The group algebra of a group G is an algebra whose representations are equivalent to representations of G.
- The group algebra of a finite group is a semi-simple, finite-dimensional algebra.
- We interpret the structure theorem for such algebras in representation theoretic terms.
- We construct an explicit isomorphism between the group algebra and a direct sum of matrix algebras and describe its inverse.
- This generalises the Fourier transform.
- The proof uses the Schur Orthogonality Relations for matrix coefficients of irreducible representations.

The group algebra

Definition

Let G be a group and K a field.
The group algebra $K[G]$ is the ring of all functions $f\colon G \to K$ with finite support and with the convolution product

$$(f_1 * f_2)(g) := \sum_{h,k \in G : hk = g} f_1(h) f_2(k) = \sum_{h \in G} f_1(h) f_2(h^{-1} g) = \sum_{h \in G} f_1(gh) f_2(h^{-1}).$$

Lemma

The group algebra $K[G]$ is a unital K-algebra.

Example

If $G = \mathbb{Z}$, then $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[t, t^{-1}]$ (Laurent polynomials).
The isomorphism maps $\delta_n \mapsto t^n$ for $n \in \mathbb{Z}$.

Representations of the group and the group algebra

Definition

A representation of a group G on a vector space V is a group homomorphism $\pi\colon G \to \mathrm{Aut}(V)$.
Here $\mathrm{Aut}(V)$ denotes the group of invertible linear maps on V.
If V is a Banach space, then we allow only bounded linear maps on V.

Proposition

If $\pi\colon G \to \mathrm{Aut}(V)$ is a group representation, then

$$\bar{\pi}(f) := \sum_{g \in G} f(g)\pi(g)$$

defines an algebra representation $\bar{\pi}\colon K[G] \to \mathrm{End}(V)$.
Conversely, any unital algebra representation comes from a unique group representation of G.

Semisimplicity

Theorem

Let G be a finite group. The group algebra $\mathbb{C}[G]$ is semi-simple. Let $\widehat{G}$ be the set of isomorphism classes of irreducible representations of G and let d_π for $\pi \in \widehat{G}$ be the dimension of the representation π. There is an isomorphism $\mathbb{C}[G] \cong \bigoplus_{\pi \in \widehat{G}} \mathbb{M}_{d_\pi}\mathbb{C}$.

Corollary

Let G be a finite group. Then $\sum_{\pi \in \widehat{G}} d_\pi^2 = |G|$.

Proposition

Let G be a group, let K be a field, and let C be the set of all finite conjugacy classes in G. The characteristic functions of $\langle g \rangle \in C$ form a basis for the centre of $K[G]$.
The number of isomorphism classes of irreducible $\mathbb{C}$-linear representations is the number of conjugacy classes in G.

An example: the symmetric group on three letters

- The group $G = S_3$ has six elements and three conjugacy classes: the trivial element and the classes of cycles of length two and three.
- So $\widehat{G}$ has three elements.
- There are two homomorphisms $G \to \{\pm 1\}$: the trivial homomorphism and the sign homomorphism that maps each transposition to -1.
- The third representation must have dimension 2.
- The group G acts on $\mathbb{C}^3$ by permuting the basis vectors. $\mathbb{C} \cdot (1, 1, 1)$ is an invariant subspace.
- The orthogonal complement is the 2-dimensional, irreducible representation of G.

The Fourier transform for finite Abelian groups

Theorem

Let G be a finite Abelian group.
Then $\widehat{G}$ is equal to the set of characters of G.
The Fourier transform is an isomorphism between
the algebra $\mathbb{C}[G]$ with the convolution product and
the algebra $\mathbb{C}[\widehat{G}]$ with pointwise multiplication.
Let G be a finite cyclic group. So $G = \mathbb{Z}/n$ for some $n \in \mathbb{N}_{\geq 1}$.
Any character of G is $\chi_l \colon k \mapsto \exp(2\pi \mathrm{i} kl/n)$ for some $l \in \mathbb{Z}/n$.
So $\widehat{G} \cong \mathbb{Z}/n$.

The Fourier transform for finite groups

Definition

Any representation (V, π) of G yields a unital algebra homomorphism $\bar{\pi}\colon \mathbb{C}[G] \to \mathrm{End}(V) \cong \mathbb{M}_{d_\pi}\mathbb{C}$, where $d_\pi := \dim V$. Letting (V, π) run through the set $\widehat{G}$ of all irreducible representations, we get a unital algebra homomorphism

$$F\colon \mathbb{C}[G] \to \prod_{\pi \in \widehat{G}} \mathbb{M}_{d_\pi}\mathbb{C}.$$

This is called the Fourier transform for G.

Theorem

The Fourier transform is invertible. Its inverse $F^{-1}\colon \bigoplus_{\pi \in \widehat{G}} \mathrm{End}(V_\pi) \to \mathbb{C}[G]$ *is given by*

$$F^{-1}\big((x_\pi)_{\pi \in \widehat{G}}\big)(g) := \sum_{\pi \in \widehat{G}} \frac{d_\pi}{|G|} \operatorname{tr}(x_\pi \circ \pi_{g^{-1}}).$$

Invariant inner products

Proposition

Let $\pi\colon G \to \mathrm{Aut}(V)$ *be a group representation of* G *on a finite-dimensional vector space* V*.*
There is a G*-invariant inner product on* V*, that is, an inner product with* $\langle \pi_g v \mid \pi_g w \rangle = \langle v \mid w \rangle$ *for all* $g \in G$*,* $v, w \in V$*.*

Corollary

Any finite-dimensional representation V *of* G *is a direct sum of irreducible representations.*
The canonical map $F\colon \mathbb{C}[G] \to \bigoplus_{\pi \in \widehat{G}} \mathrm{End}(V_\pi)$ *is injective.*

An adjoint for F

Definition

We define an inner product on $\mathrm{End}(V_\pi)$ for $\pi \in \widehat{G}$ by

$$\langle x \mid y \rangle := \frac{d_\pi}{|G|} \cdot \mathrm{tr}(x^* y).$$

for all $x, y \in \mathrm{End}(V_\pi)$.
We give $\mathbb{C}[G]$ the usual inner product as in $\ell^2(G)$.

As we shall see, the normalisation factor $d_\pi / |G|$ makes the Fourier transform $F\colon \mathbb{C}[G] \to \bigoplus_{\pi \in \widehat{G}} \mathbb{M}_{d_\pi}\mathbb{C}$ isometric.

Lemma

$$F^*\big((x_\pi)_{\pi \in \widehat{G}}\big)(g) = \sum_{\pi \in \widehat{G}} \frac{d_\pi}{|G|} \,\mathrm{tr}(x_\pi \circ \pi_{g^{-1}}).$$

Matrix coefficients

Definition

For $v, w \in V$, define $c_{v,w}(g) := \langle \pi_g w \mid v \rangle$. The function $c_{v,w}$ is called a matrix coefficient of the representation π.

- There is a canonical vector space isomorphism

$$V \otimes \overline{V} \to \mathrm{End}(V), \qquad v \otimes w \mapsto |v\rangle\langle w|.$$

- So there is a linear map

$$C\colon \mathrm{End}(V) \to \mathbb{C}[G], \qquad C\left(|v\rangle\langle w|\right) = c_{v,w} := \langle \pi_g w \mid v \rangle.$$

- $C(x)(g) = \mathrm{tr}(x \circ \pi_{g^{-1}})$.

Canonical intertwiners to the regular representation

$$|v\rangle\!\rangle\colon \mathbb{C}[G] \to V, \qquad |v\rangle\!\rangle(f) := \sum_{g\in G} f(g)\pi_g(v),$$

$$\langle\!\langle v|\colon V \to \mathbb{C}[G], \qquad \langle\!\langle v|(w)(g) := c_{w,v} = \langle \pi_g(v) \mid w\rangle.$$

Lemma

$|v\rangle\!\rangle^* = \langle\!\langle v|$.
$\langle\!\langle v|$ and hence $|v\rangle\!\rangle$ are G-equivariant:
$\langle\!\langle v| \circ \pi_g = \lambda_g \circ \langle\!\langle v|$ and $|v\rangle\!\rangle \circ \lambda_g = \pi_g \circ |v\rangle\!\rangle$.

Lemma (Schur)

Let (V, π) and (W, ρ) be finite-dimensional irreducible rep's of G.
Let $T\colon V \to W$ be G-equivariant. Then T is invertible or zero.
If $(V, \pi) = (W, \rho)$, then T is a scalar multiple of the identity:
$T = a \cdot \mathrm{Id}_V$ for some $a \in \mathbb{C}$.

Orthogonality Relations

Theorem

Let G be a finite group. Let (V, π) and (W, ρ) be irreducible representations of G with G-invariant inner products. Let $v_1, v_2 \in V$, $w_1, w_2 \in W$.
If (V, π) and (W, ρ) are not isomorphic, then $c_{v_1,v_2} \perp c_{w_1,w_2}$ in $\mathbb{C}[G]$, that is,

$$0 = \sum_{g \in G} \overline{\langle \pi_g v_1 \mid v_2 \rangle} \langle \rho_g w_1 \mid w_2 \rangle = \sum_{g \in G} \langle v_2 \mid \pi_g v_1 \rangle \langle \rho_g w_1 \mid w_2 \rangle.$$

If $(V, \pi) = (W, \rho)$ with the same inner product, then

$$\langle c_{v_1,v_2} \mid c_{w_1,w_2} \rangle = \frac{|G|}{d_\pi} \langle w_1 \mid v_1 \rangle \cdot \langle v_2 \mid w_2 \rangle.$$

Corollary

The map F^* is isometric.