Noncommutative Geometry IV: Differential Geometry

7. Category algebras and quiver algebras

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Category algebras and quiver algebras

- We associate noncommutative algebras to categories and quivers.
- Special cases of these constructions are
  - matrix algebras;
  - algebras of upper triangular matrices;
  - group algebras.
- For finite categories and quivers, we describe the nilradical and the semisimple quotient of these algebras.

Definition of category algebras

Definition

Let K be a field, $\mathcal{C}$ a small category, $\cdot$ its composition,
$\mathcal{C}$ or $\mathcal{C}^{(1)}$ its morphism space, $\mathcal{C}^{(0)}$ its object space.
Let $K[\mathcal{C}]$ be the free K-vector space over $\mathcal{C}^{(1)}$; basis: $(\delta_f)_{f\in\mathcal{C}^{(1)}}$.
$K[\mathcal{C}] \cong \{f\colon \mathcal{C} \to K : \operatorname{supp} f \text{ finite}\}$.
We define the multiplication on basis vectors of $K[\mathcal{C}]$ by

$$\delta_f * \delta_g := \begin{cases} \delta_{f\cdot g} & \text{if } f \cdot g \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

This extends to a unique bilinear map

$$*\colon K[\mathcal{C}] \times K[\mathcal{C}] \to K[\mathcal{C}], \qquad f * g(\alpha) := \sum_{\beta\cdot\gamma=\alpha} f(\beta)\cdot g(\gamma).$$

and turns $K[\mathcal{C}]$ into a K-algebra.

The algebra $K[\mathcal{C}]$ is finite-dimensional if and only if $\mathcal{C}$ is finite.

Universal property of the category algebra

Definition

Let $\mathcal{C}$ be a small category. A representation of $\mathcal{C}$ consists of

- vector spaces V_x for $x \in \mathcal{C}^{(0)}$, and
- arrows $\pi_f \colon V_x \to V_y$ for arrows $f \colon x \to y$ in $\mathcal{C}$,

such that $\pi_f \pi_g = \pi_{fg}$ if f, g are composable in $\mathcal{C}$.

Theorem

Let $\mathcal{C}$ be a small category. The category of K-linear representations of $\mathcal{C}$ is equivalent to the category of non-degenerate $K[\mathcal{C}]$-modules.

Some examples

Example

Let G be a group. View G as a category with one object.
The category algebra is the same as the group algebra of G.

Example

View a set X as a category with only identical morphisms:
$\mathcal{C}^{(0)} = \mathcal{C}^{(1)} = X$.
Then $\mathbb{C}[X]$ carries the pointwise multiplication: $\mathbb{C}[X] \cong \bigoplus_{x \in X} \mathbb{C}$.

Example

Let $\mathcal{C}$ be the category with two objects 1 and 2 and only one non-identity morphism f from 1 to 2. Its category algebra is isomorphic to the subalgebra of lower triangular matrices via

$$\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{11}\delta_{\mathrm{Id}_1} + a_{21}\delta_f + a_{22}\delta_{\mathrm{Id}_2}.$$

The full matrix algebra as a category algebra

Example

Let $X_n = \{1, 2 \ldots, n\}$ and let $\mathcal{C}_n$ be the category with exactly one morphism $i \to j$ for each $i, j \in X_n$. So $\mathcal{C}_n^{(1)} = X_n \times X_n$.
We identify $K[\mathcal{C}_n]$ with the space of functions $X_n \times X_n \to K$.
Let $\delta_{i,j}$ be the basis vector for the unique morphism $j \to i$ in $\mathcal{C}_n$.
The multiplication in $K[\mathcal{C}_n]$ is $(f * g)(i, j) = \sum_{l=1}^{n} f(i, l)g(l, j)$.
This is matrix multiplication. Hence $K[\mathcal{C}_n] \cong \mathbb{M}_n K$.

Quivers

Definition

A quiver is a directed graph. It has a set of objects Q^0, a set of arrows Q^1 with range and source maps $Q^1 \rightrightarrows Q^0$ and no further structure.
Thus "quiver" is a synonym for "directed graph."
A path in a quiver is a finite sequence of composable arrows.
There is an "empty path" v starting and ending at $v \in Q^0$.
A loop is a path with the same head and tail.
The paths in a quiver form a category with respect to concatenation of paths, called its path category.

The path category is finite if and only if the quiver is finite and has no (directed) loops.

Proposition

Let $\mathcal{C}$ be the path category of a finite quiver without loops. The nilradical $\operatorname{rad} K[\mathcal{C}]$ *is the linear span of δ_α for the non-empty paths α in $\mathcal{C}$. $K[\mathcal{C}]/\operatorname{rad} K[\mathcal{C}] \cong \bigoplus_{x\in\mathcal{C}^{(0)}} \mathbb{C}$.*