Noncommutative Geometry IV: Differential Geometry

8. The group algebra of the dihedral group

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The group algebra of the dihedral group

- The infinite dihedral group D_∞ is generated by two elements s, t with the relations $s^2 = t^2 = 1$.
- There is an isomorphism between $\mathbb{C}[D_\infty]$ and $\left\{ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in \mathbb{M}_2\mathbb{C}[x] : f_{21}(0) = f_{21}(1) = 0 \right\}$.
- Thus $\mathbb{C}[D_\infty]$ has centre isomorphic to $\mathbb{C}[x]$, and it is finitely generated as a module over the centre.
- For a unital algebra A with this property, the map $\operatorname{Prim}(A) \to \widehat{A}$ is bijective, and all primitive ideals of A are maximal. There is a finite-to-one map $\widehat{A} \to \widehat{Z(A)}$.

Definition

The infinite dihedral group D_∞ is the group of affine transformations of $\mathbb{R}$ that is generated by translations τ_n for $n \in \mathbb{Z}$ and the reflection s at the origin.

- An element of D_∞ is either a translation τ_n or a reflection $\tau_n \circ s$ for some unique $n \in \mathbb{Z}$.
- The multiplication table is determined by $\tau_n \tau_m = \tau_{n+m}$, $s^2 = 1$, and $s\tau_n = \tau_{-n}s$.
- The subgroup of translations $\mathbb{Z} \cong \{\tau_n\}$ in D_∞ is normal, and $D_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ acts on $\mathbb{Z}$ by $n \mapsto -n$.

Other generators

- The group D_∞ is generated by s and $t := \tau_1 \circ s$ because $ts = \tau_1$, $st = \tau_{-1}$.
- Both s and t are reflections.
  They satisfy no relations besides $s^2 = t^2 = 1$.
- A representation of D_∞ is equivalent to a pair of linear operators S and T with $S^2 = T^2 = 1$.
- $p := \frac{1}{2}(1+s)$, $q := \frac{1}{2}(1+t)$.
- The relations $s^2 = t^2 = 1$ are equivalent to $p^2 = p$ and $q^2 = q$.
- A representation of D_∞ is equivalent to a pair of idempotent operators P and Q.

Representations in dimensions 1 and 2

- Representations of dimension 1 are characters.
- A character on D_∞ is given by two arbitrary signs $\chi(s), \chi(t) \in \{\pm 1\}$.
- Thus there are four 1-dimensional representations.

Proposition

There is a 1-parameter family of 2-dimensional representations with

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 2x-1 & 2 \\ 2x(1-x) & 1-2x \end{pmatrix}.$$

It is irreducible if $x \neq 0, 1$.
Any irreducible 2-dimensional representation is equivalent to exactly one of these.

The group algebra

Proposition

The homomorphism $\rho\colon \mathbb{C}[D_\infty] \to \mathbb{M}_2\mathbb{C}[x]$ *that maps*

$$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 2x-1 & 2 \\ 2x(1-x) & 1-2x \end{pmatrix}$$

is injective.
Its range is the subalgebra

$$\left\{ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} : f_{21}(0) = f_{21}(1) = 0 \right\}.$$

Corollary

The centre of $\mathbb{C}[D_\infty]$ *is isomorphic to* $\mathbb{C}[x]$*; the isomorphism maps* x *to* $pqp + (1-p)(1-q)(1-p) = 1 + (st+ts)/2$*.*

The representation theory of the dihedral group

Theorem (Schur's Lemma)

Let A be a $\mathbb{C}$-algebra of at most countable dimension. Let (V, f) be an irreducible representation of A. Let $T : V \to V$ be an A-module homomorphism, that is, a $\mathbb{C}$-linear map that commutes with $f(A)$. Then $T = \kappa \cdot \mathrm{Id}_V$ for some $\kappa \in \mathbb{C}$.

Corollary

Any irreducible representation of $A := \mathbb{C}[D_\infty]$ is of dimension at most 2. The canonical map $\widehat{A} \to \mathrm{Prim}(A)$ is bijective and all primitive ideals are maximal.
The set of equivalence classes of irreducible representations is in bijection with $\mathbb{C} \setminus \{0, 1\} \sqcup \mathbb{Z}/2 \times \mathbb{Z}/2$, where $y \in \mathbb{C} \setminus \{0, 1\}$ corresponds to the representation ρ_y and points (n, m) in $\{0, 1\} \times \{0, 1\}$ correspond to the characters given by $s \mapsto (-1)^n$, $t \mapsto (-1)^m$.

Algebras of finite type

Definition

A unital $\mathbb{C}$-algebra is called finite type if
it is finitely generated as a module over its centre.
That is, there are finitely many elements $x_1, \ldots, x_n \in A$ such that every element of A can be written as $x_1 z_1 + \cdots + x_n z_n$ with central elements $z_1, \ldots, z_n \in Z(A)$.

Theorem

Let A be a unital algebra of finite type over $\mathbb{C}$. Let $Z(A)$ be its centre. Assume that A has a countable basis over $\mathbb{C}$.
Then the map $\mathrm{Prim}(A) \to \widehat{A}$ is bijective and all primitive ideals of A are maximal.
Each irreducible representation of A restricts to a character on $Z(A)$. The resulting map $\widehat{A} \to \widehat{Z(A)}$ is finite-to-one.