Noncommutative Geometry IV: Differential Geometry

9. Involutive algebras

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Involutive algebras

- A *-algebra is a $\mathbb{C}$-algebra with a conjugate-linear involution.
- For finite-dimensional algebras, this is related to semi-simplicity.
- We examine some examples of *-algebras.
- We define new *-algebras through universal properties: the Toeplitz algebra and Leavitt path algebras.

Involutive algebras – the definition

Definition

An involutive algebra or *-algebra is a $\mathbb{C}$-algebra A with a map $A \to A$, $a \mapsto a^*$, that is

$$
\begin{aligned}
\text{conjugate-linear} \quad & (\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^* \\
\text{involutive} \quad & (a^*)^* = a \\
\text{anti-multiplicative} \quad & (a \cdot b)^* = b^* \cdot a^*
\end{aligned}
$$

Definition

A *-homomorphism between two *-algebras A and B is an algebra homomorphism $\varphi\colon A \to B$ with $\varphi(a^*) = \varphi(a)^*$.

Examples of involutive algebras

Example

$\mathbb{M}_n\mathbb{C}$ for $n \in \mathbb{N} \cup \{\infty\}$ with $(x_{ij})^* := (\overline{x_{ji}})$ is a *-algebra.

Example

Let G be a group. Define $f^*(g) := \overline{f(g^{-1})}$ for $f \in \mathbb{C}[G]$, $g \in G$. This makes $\mathbb{C}[G]$ a *-algebra.

Example

Let X be a manifold. Then $f^*(x) := \overline{f(x)}$ for $f \in \mathrm{C}^\infty(X)$, $x \in X$, makes $\mathrm{C}^\infty(X)$ a *-algebra.

Example

Let $\mathcal{H}$ be a Hilbert space and let $\mathbb{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. If $x \in \mathbb{B}(\mathcal{H})$, then there is $x^* \in \mathbb{B}(\mathcal{H})$ with $\langle xv \mid w \rangle = \langle v \mid x^* w \rangle$ for all $v, w \in \mathcal{H}$.

This makes $\mathbb{B}(\mathcal{H})$ a *-algebra.

Special elements in *-algebras

Definition

Let A be a *-algebra. An element $a \in A$ is called

self-adjoint	$a^* = a$
projection	$a^* = a$ and $a^2 = a$
partial isometry	$aa^*a = a$ or, equivalently, $a^*aa^* = a^*$.
unitary	$aa^* = a^*a = 1$ (needs unital A)
isometry	$a^*a = 1$
coisometry	$aa^* = 1$.

If a is a partial isometry, then aa^* and a^*a are projections. They are called the range and source projections of a.

Universal *-algebra for a self-adjoint element

Lemma

Give $\mathbb{C}[x]$ the involution defined by $x^* = x$.
For any self-adjoint element a in a unital *-algebra A,
there is a unique unital *-homomorphism $f : \mathbb{C}[x] \to A$ with $f(x) = a$.
We call $x \in \mathbb{C}[x]$ the universal self-adjoint element of a unital *-algebra
and we call $\mathbb{C}[x]$ the universal unital *-algebra generated by a self-adjoint element.

More universal *-algebras

Lemma

The projection $(1, 0)$ *in the algebra* $\mathbb{C} \oplus \mathbb{C}$ *with the involution* $(x, y)^* := (\overline{x}, \overline{y})$ *is the universal projection in a unital* **-algebra. That is, unital* **-homomorphisms* $\mathbb{C} \oplus \mathbb{C} \to A$ *correspond to projections in* A *by evaluation at* $(1, 0)$*.*

Proposition

Equip $\mathbb{C}[t, t^{-1}] \cong \mathbb{C}[\mathbb{Z}]$ *with the involution* $\left(\sum_{n\in\mathbb{Z}} c_n t^n\right)^* := \sum_{n\in\mathbb{Z}} \overline{c_n} t^{-n} = \sum_{n\in\mathbb{Z}} \overline{c_{-n}} t^n$*.*
The element $t \in \mathbb{C}[t, t^{-1}]$ *is the universal unitary element of a unital* **-algebra.*

The Toeplitz algebra

Definition

The unilateral shift is the operator S on the Hilbert space $\ell_2(\mathbb{N})$ that shifts every basis vector one to the right: $S\delta_n := \delta_{n+1}$ for all $n \in \mathbb{N}$.
The Toeplitz algebra $\mathcal{T}$ is the *-subalgebra of $\mathbb{B}(\ell_2\mathbb{N})$ generated by S.

Theorem

The isometry $S \in \mathcal{T}$ is the universal isometry in a unital *-algebra. For any isometry s in a unital *-algebra A, there is a unique unital *-homomorphism $\varrho\colon \mathcal{T} \to A$ with $\varrho(S) = s$.

Structure of the Toeplitz algebra

Definition

Let $I \subseteq A$ be an ideal in an algebra A.
Let A/I be the quotient algebra and let $i\colon I \to A$ and $p\colon A \to A/I$ be the canonical maps. We call the diagram

$$I \xrightarrow{i} A \xrightarrow{p} A/I$$

an algebra extension and we call A an extension of A/I by I.

Theorem

The Toeplitz algebra is an extension of the algebra $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials by the algebra $\mathbb{M}_\infty\mathbb{C}$ of finite matrices.

Representation theory of the Toeplitz algebra

Proposition

Let $W \subseteq \ell_2\mathbb{N}$ be the algebraic linear span of the basis vectors $(\delta_n)_{n\in\mathbb{N}}$ and represent $\mathcal{T}$ on W by $S \mapsto S|_W$. This representation is irreducible and faithful, so that $\mathcal{T}$ is a primitive algebra.

Proposition

Any irreducible representation of $\mathcal{T}$ that is non-zero on $\mathbb{M}_\infty\mathbb{C}$ is isomorphic to the representation above.
The other irreducible representations of $\mathcal{T}$ are characters on the quotient $\mathcal{T}/\mathbb{M}_\infty\mathbb{C} \cong \mathbb{C}[t, t^{-1}]$.
The canonical map $\hat{\mathcal{T}} \to \mathrm{Prim}(\mathcal{T})$ is a bijection.

Theorem

Any isometry on a Hilbert space can be decomposed as a direct sum of a unitary and of copies of the unilateral shift.

Leavitt path algebras

Definition

The Leavitt path algebra $L(\Gamma, \Gamma'_0)$ of a directed graph Γ relative to a subset Γ'_0 of suitable vertices is the universal *-algebra with generators S_v for $v \in \Gamma_0$ and S_α for $\alpha \in \Gamma_1$, subject to the following relations:

(V) $S_v \cdot S_w = 0$ for $v \neq w$ and $S_v^2 = S_v = S_v^*$;

(E) $S_\alpha S_{s(\alpha)} = S_\alpha$ and $S_{r(\alpha)} S_\alpha = S_\alpha$ for all α;

(CK1) $S_\alpha^* S_\beta = 0$ if $\alpha \neq \beta$ and $S_\alpha^* S_\alpha = S_{s(\alpha)}$;

(CK2) $\sum_{\alpha \in r^{-1}(v)} S_\alpha S_\alpha^* = S_v$ for all $v \in \Gamma'_0$.

Proposition

Let Γ have two vertices a, b and two edges $v\colon a \to b$ and $w\colon b \to b$. Let $\Gamma'_0 = \{b\}$. The Leavitt path algebra $L(\Gamma, \Gamma'_0)$ is isomorphic as a *-algebra to the Toeplitz algebra $\mathcal{T}$.