Noncommutative Geometry IV: Differential Geometry

10. Crossed products

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Crossed products

- A crossed product algebra is built from an action of a group G on an algebra A by automorphisms.
- Its representations are equivalent to covariant representations of the group G and the algebra A.
- The group algebra of a semi-direct product of groups is a crossed product.
- We compute some examples of crossed products.
- For an action of a finite group on $C^{\infty}(X)$, we describe the representation theory of the crossed product.

The definition of the crossed product

Convolution

The data

G a group

A an algebra over some field

α action of G on A by automorphisms

Definition

The crossed product $G \ltimes_\alpha A = A \rtimes_\alpha G$ is the vector space of functions $G \to A$ with finite support, equipped with the convolution product

$$(f_1 * f_2)(g) := \sum_{h \in G} f_1(h) \cdot \alpha_h\big(f_2(h^{-1}g)\big).$$

The definition of the crossed product

Generators and relations

Define $a\delta_g \in G \ltimes_\alpha A$ for $g \in G$, $a \in A$ by
$a\delta_g(h) = 0$ for $h \neq g$ and $a\delta_g(g) = a$.
Every element of $G \ltimes A$ decomposes uniquely as a sum
$\sum_{g \in F} a(g)\delta_g$ for some finite subset $F \subseteq G$.
The convolution product is generated by the rule

$$a\delta_g * b\delta_h = a\alpha_g(b)\delta_{gh}.$$

Covariant representations

Definition

A covariant representation of (A, G, α) is a pair of representations $f \colon A \to \mathrm{End}(V)$, $U \colon G \to \mathrm{Aut}(V)$ on the same vector space V that satisfy the covariance condition $U_g f(a) U_g^{-1} = f(\alpha_g(a))$ for all $g \in G$, $a \in A$.

Proposition

Let A be a unital algebra, let G be a group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a group homomorphism.
The category of unital representations of $A \rtimes_\alpha G$ is isomorphic to the category of unital covariant representations of (A, G, α).

Example

Let $\lambda \colon A \to \mathrm{End}(A)$ be the left regular representation, $\lambda_a(b) := a \cdot b$. The pair (λ, α) is a covariant representation:
$\alpha_g \lambda_a \alpha_{g^{-1}}(b) = \alpha_g(a \cdot \alpha_{g^{-1}}(b)) = \alpha_g(a) b = \lambda_{\alpha_g(a)}(b)$.

Semi-direct products of groups

Let N and H be groups.
Write $\mathrm{Aut}(N)$ for the group of group automorphisms of N.
Let $\alpha\colon H \to \mathrm{Aut}(N)$ be a group homomorphism.

Definition

The semi-direct product group $H \ltimes_\alpha N = N \rtimes_\alpha H$ is $N \times H$ with the product $(n_1, h_1) \cdot (n_2, h_2) := (n_1 \alpha_{h_1}(n_2), h_1 h_2)$.

Example

The isometry group of $\mathbb{R}^n$ is the semi-direct product of the group $N = \mathbb{R}^n$ of translations and the orthogonal group $H = \mathrm{O}(n)$ with for the canonical action $\alpha\colon \mathrm{O}(n) \to \mathrm{Aut}(\mathbb{R}^n)$.

Example

The infinite dihedral group is a semi-direct product $D_\infty = \mathbb{Z} \rtimes_\alpha \mathbb{Z}/2$, where $\alpha_n(m) = (-1)^n \cdot m$ for $m \in \mathbb{Z}$, $n \in \mathbb{Z}/2$.

Crossed products and semi-direct product groups

Lemma

Let N and H be groups and $\alpha\colon H \to \mathrm{Aut}(N)$.
This induces an action $\bar{\alpha}\colon H \to \mathrm{Aut}(K[N])$, $\bar{\alpha}(\delta_n) := \delta_{\bar{\alpha}(n)}$.
And $K[N] \rtimes_{\bar{\alpha}} H \cong K[N \rtimes_\alpha H]$.

In particular,

$$K[D_\infty] \cong K[\mathbb{Z}] \rtimes \mathbb{Z}/2 \cong K[t, t^{-1}] \rtimes \mathbb{Z}/2.$$

Some crossed product computations

Proposition

Let A be a unital $\mathbb{C}$-algebra, G a finite group, and $\alpha\colon G \to \mathrm{Aut}(A)$. Let $\lambda_g f(x) := f(g^{-1}x)$ denote the left regular representation of G on $\mathbb{C}[G]$. Let G act on $B := A \otimes \mathrm{End}(\mathbb{C}[G])$ by
$g \cdot (a \otimes x) := \alpha_g(a) \otimes \lambda_g x \lambda_g^{-1}$.
Then $A \rtimes_\alpha G$ is naturally isomorphic to the fixed point subalgebra of the G-action on B.

Lemma

Let G be a countably infinite group. Let $A = \mathbb{C}[G]$ with pointwise multiplication and let $\alpha = \lambda$ be the left regular representation. Then $A \rtimes_\alpha G \cong \mathbb{M}_\infty\mathbb{C}$.

Lemma

$G \ltimes \mathbb{C}[G/H] \cong \mathbb{C}[H] \otimes \mathbb{M}_{|G/H|}\mathbb{C}$.
The irreducible representations of $G \ltimes \mathbb{C}[G/H]$ are in bijection with irreducible representations of H.

Representation theory

Let X be a smooth compact manifold, G a finite group.
An action α of G on X by diffeomorphisms induces an action of G on $C^\infty(X)$ by algebra automorphisms.

A $C^\infty(X) \rtimes_\alpha G$

$G\backslash X$ orbit space;

G_x stabiliser subgroup of $x \in X$

$\hat{G}_x$ set of irreducible representations of G_x.

Theorem

There is a canonical bijection between $\hat{A}$ and $\bigsqcup_{x \in G\backslash X} \hat{G}_x$.
The map $\hat{A} \to \mathrm{Prim}(A)$ is bijective and any primitive ideal in A is maximal and closed in the natural topology.