Noncommutative Geometry IV: Differential Geometry

11. Morita equivalence

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Morita equivalence

- We are going to motivate the importance of Morita equivalence for noncommutative geometry.
- In many geometric situations, the noncommutative algebra describing it is only unique up to Morita equivalence.
- Morita equivalence is defined as an equivalence between the module categories.
- We characterise it through bimodules over the algebras with certain properties and using corners defined by full projections.
- We provide some important examples of Morita equivalent algebras.

Quotient spaces in noncommutative geometry

X smooth compact manifold

G a group

α action of G on X by diffeomorphisms

basic paradigm of noncommutative geometry

Replace the orbit space $G\backslash X$
by the noncommutative algebra $C^\infty(X) \rtimes_\alpha G$.
This is particularly interesting for infinite groups G,
where the orbit space $G\backslash X$ is usually very badly behaved.

Irrational rotations

Example

Fix $\vartheta \in \mathbb{R}$.
Let $X := \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be a circle and let $G := \mathbb{Z}$ act on X by rotations: $n \bullet z := \exp(2\pi i \vartheta n) \cdot z$ for all $n \in \mathbb{Z}$.
If ϑ is rational, $\vartheta = p/q$, then $q \bullet z = z$ for all $z \in X$, so that the action of G factors through an action of the finite group $\mathbb{Z}/q\mathbb{Z}$.
Hence $G\backslash X = (\mathbb{Z}/q\mathbb{Z})\backslash X$ is again a circle.
If ϑ is irrational, then the orbit $Gz := \{n \bullet z : n \in \mathbb{Z}\}$ is dense in X for each $z \in X$. Hence any G-invariant continuous function on X is constant. The orbit space $G\backslash X$ carries no useful topology and is certainly not a smooth manifold.

Correspondence principle

The rotations with angles $2\pi\vartheta$ and $2\pi/\vartheta$ generate "the same" orbit space

$$\mathbb{T}/2\pi\vartheta\mathbb{Z} \cong \mathbb{R}/2\pi(\mathbb{Z}+\vartheta\mathbb{Z}) \xrightarrow[x\mapsto\vartheta^{-1}x]{\cong} \mathbb{R}/2\pi(\vartheta^{-1}\mathbb{Z}+\mathbb{Z}) \cong \mathbb{T}/2\pi\vartheta^{-1}\mathbb{Z}.$$

Therefore, noncommutative geometry should not distinguish between the crossed products $C^\infty(\mathbb{T}) \rtimes_\vartheta \mathbb{Z}$ and $C^\infty(\mathbb{T}) \rtimes_{\vartheta^{-1}} \mathbb{Z}$.
If a finite group G acts freely on a smooth manifold X, then noncommutative geometry should not distinguish between the algebras $C^\infty(X) \rtimes_\alpha G$ and $C^\infty(G\backslash X)$.

Morita equivalence

Definition

Let R be a (unital) ring. Let $\mathfrak{Mod}_R$ be the category with left R-modules as objects, module homomorphisms as arrows, and the usual composition.

Two rings R and S are Morita equivalent if $\mathfrak{Mod}_R$ and $\mathfrak{Mod}_S$ are equivalent categories.

Theorem

Let M be a smooth compact manifold and
let α be a free group action of a finite group G on M.
Then $\mathrm{C}^\infty(M) \rtimes_\alpha G$ and $\mathrm{C}^\infty(G\backslash M)$ are Morita equivalent.

Bimodule tensor product

Definition

Let R and S be two rings, Q an S, R-bimodule, M an R-module.
The R-balanced tensor product $Q \otimes_R M$ is the quotient of $Q \otimes M$ by the subgroup generated by $q \cdot r \otimes m - q \otimes r \cdot m$.
The group $Q \otimes_R M$ carries a unique S-module structure with $s \cdot (q \otimes m) := (s \cdot q) \otimes m$ all $s \in S$, $q \in Q$, $m \in M$.
If M is an R, T-module for a third ring T,
then $Q \otimes_R M$ carries a unique right T-module structure with $(q \otimes m) \cdot t := q \otimes (m \cdot t)$.
This makes $Q \otimes_R M$ an S, T-bimodule.

Theorem

Two rings R and S are Morita equivalent if and only if there are an S, R-bimodule Q and an R, S-bimodule P with bimodule isomorphisms $Q \otimes_R P \cong S$ and $P \otimes_S Q \cong R$.

Some examples of Morita equivalence

Example

Let R be a unital ring. Then R is Morita equivalent to $\mathbb{M}_n(R)$.
The bimodules are $P = R^n$ and $Q = R^n$ with matrix-vector multiplication.
Matrix multiplication between $1 \times n$- and $n \times 1$-matrices gives the bimodule isomorphisms $Q \otimes_R P \cong S$ and $P \otimes_S Q \cong R$.

Definition

An idempotent element $p \in A$ in a ring A is called full if
it generates A as an ideal, that is,
elements of the form apb with $a, b \in A$ span A.

Theorem

Let $p \in A$ be a full idempotent in a unital ring A.
Then A is Morita equivalent to pAp.

The linking ring

Definition

Let A and B be two unital rings. Let P and Q be an A, B-bimodule and a B, A-bimodule. Let $\mu_{PQ}\colon P \otimes_B Q \to A$ and $\mu_{QP}\colon Q \otimes_A P \to B$ be bimodule homomorphisms. Assume also that

$\mu_{PQ} \otimes_A \mathrm{Id}_P = \mathrm{Id}_P \otimes_B \mu_{QP}\colon P \otimes_B Q \otimes_A P \to P$ and

$\mu_{QP} \otimes_B \mathrm{Id}_Q = \mathrm{Id}_Q \otimes_A \mu_{PQ}\colon Q \otimes_A P \otimes_B Q \to Q$.

The linking ring associated to $(A, B, P, Q, \mu_{PQ}, \mu_{QP})$ is the unital ring with underlying vector space $L := A \oplus P \oplus Q \oplus B$ and with the associative multiplication

$$\begin{pmatrix} a_1 & p_1 \\ q_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & p_2 \\ q_2 & b_2 \end{pmatrix} := \begin{pmatrix} a_1 \cdot a_2 + \mu_{PQ}[p_1 \otimes q_2] & a_1 \cdot p_2 + p_1 \cdot b_2 \\ q_1 \cdot a_2 + b_1 \cdot q_2 & \mu_{QP}[q_1 \otimes b_2] + b_1 \cdot b_2 \end{pmatrix}$$

Morita equivalence through linking ring

Proposition

Let A and B be Morita equivalent. Then there are bimodules P and Q and bimodule homomorphisms μ_{PQ} and μ_{QP} as above, so that the linking ring L is defined.
The elements

$$p_A := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_B := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in L are idempotent elements with $p_A L p_A \cong A$ and $p_B L p_B \cong B$.
The idempotent elements p_A and p_B are full if μ_{PQ} and μ_{QP} are bimodule isomorphisms.
Two rings are Morita equivalent if and only if they are both isomorphic to full corners in the same ring.

Morita invariance of the spectrum

Proposition

Let A and B be Morita equivalent unital rings. Then
$\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$ *and* $\hat{A} \cong \hat{B}$.
The lattices $\mathbb{I}(A)$ *and* $\mathbb{I}(B)$ *of ideals in A and B are isomorphic.*