Noncommutative Geometry IV: Differential Geometry

12. Derivations as noncommutative differentiation

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Derivations as noncommutative differentiation

- We now start with noncommutative differential geometry.
- We describe differentiation of smooth functions on a manifold using tangent vectors and vector fields.
- We understand this algebraically using the Leibniz rule.
- This leads to the definition of a derivation into a bimodule.
- This generalises tangent vectors and vector fields for algebras of smooth functions.
- We define a subclass of inner derivations and show that all derivations for a matrix algebra are inner.
- This uses a description of derivations through sections of module extensions.
- To illustrate this theory, we discuss two-dimensional representations of algebras of smooth functions.

Differentiation of smooth functions?

- Partial derivatives of functions on $\mathbb{R}^n$ depend on coordinates.
- The coordinate-independent formulation on a manifold are directional derivatives for tangent vectors.
- The directional derivative for $v \in T_x M$, $x \in M$ is a linear map $\mathrm{C}^\infty(M, \mathbb{R}) \to \mathbb{R}$, $f \mapsto D_v f(x)$.
- A global version of this is the first order differential operator $\mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$, $f \mapsto X(f)$,
  given by a vector field X on M.
- A vector field is a smooth section of the tangent bundle.

The Leibniz rule

Theorem

Let $m \in M$. A linear map $l\colon C^\infty(M, \mathbb{R}) \to \mathbb{R}$ is of the form $f \mapsto Df_m(\vec{v})$ for some $\vec{v} \in T_m M$ if and only if $l(f \cdot g) = f(m) \cdot l(g) + l(f) \cdot g(m)$.

A linear map $l\colon C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ comes from a smooth vector field on M if and only if $l(f \cdot g) = l(f) \cdot g + f \cdot l(g)$.

Definition

Let A be an algebra and let M be an A-bimodule.
A derivation is a linear map $d\colon A \to M$ that satisfies the Leibniz rule: $d(a \cdot b) = a \cdot d(b) + d(a) \cdot b$ for all $a, b \in A$.
$\mathrm{Der}(A, M) = \{\text{derivations } A \to M\}$.

Example ($A = C^\infty(M, \mathbb{R})$ for a manifold M)

$\mathrm{Der}(A, A) \cong \{\text{smooth vector fields}\}$.
Let $x \in M$. Then $A/\ker \mathrm{ev}_x$ is an A-bimodule and $\mathrm{Der}(A, A/\ker \mathrm{ev}_x) \cong T_x M$. Derivations generalise both smooth vector fields and tangent vectors on manifolds.

Real versus complex

Here we use $\mathbb{R}$-valued functions on smooth manifolds in order to get ordinary tangent vectors and vector fields.
On $C^\infty(M, \mathbb{C})$, we get vectors in the complexified tangent bundle and the complex vector space generated by smooth vector fields.
This is less geometric.
Some things like integrating a vector field to a flow no longer work.

Inner derivations

Definition

Let A be an algebra, let M be an A-bimodule, and let $m \in M$.
The inner derivation ad_m is defined by
$\mathrm{ad}_m(a) = [m, a] := m \cdot a - a \cdot m$ for all $a \in A$.
$\mathrm{Inn}(A, M) := \{\text{inner derivations } A \to M\} \subseteq \mathrm{Der}(A, M)$.
The centre $Z(M)$ of M is the set of all $m \in M$ with $\mathrm{ad}_m = 0$, that is, $a \cdot m = m \cdot a$ for all $a \in A$.

There is a linear map

$$\mathrm{ad}\colon M \to \mathrm{Der}(A, M).$$

Its kernel is equal to the centre $Z(M)$ of M.

Theorem

Let $A = \mathbb{M}_n\mathbb{C}$. Then any derivation in $\mathrm{Der}(A, A)$ is inner.
Any derivation $A \to M$ for a unital A-bimodule M is inner.

Derivations are not Morita invariant

$\mathrm{Der}(\mathbb{M}_n\mathbb{C}, \mathbb{M}_n\mathbb{C})$ is much bigger than $\mathrm{Der}(\mathbb{C}, \mathbb{C}) = \{0\}$.

Derivations and module extensions

- Let $V \overset{i}{\rightarrowtail} X \overset{p}{\twoheadrightarrow} W$ be an extension of left A-modules:
  the map $i\colon V \to X$ is injective
  and p induces an isomorphism $X/i(V) \cong W$.
- There is a linear map – not module homomorphism –
  $s\colon W \to X$ with $p \circ s = \mathrm{Id}_W$.
- Use s to split $X \cong V \oplus W$ as a vector space.
- For $a \in A$, the operator μ_a^X of multiplication by a has the form

$$\mu_a^X = \begin{pmatrix} \mu_a^V & d_a \\ 0 & \mu_a^W \end{pmatrix}$$

  where μ_a^V and μ_a^W multiply by a on V and W and
  $d_a\colon W \to V$. This gives a derivation $d\colon A \to \mathrm{Hom}(W, V)$
  for the bimodule structure on $\mathrm{Hom}(W, V)$ defined by
  $a \cdot x \cdot b := \mu_a^V \circ x \circ \mu_b^W$ for $a, b \in A$, $x\colon W \to V$.
- Dependence on the section s: may replace s by $s + \delta$
  for some linear map $\delta\colon W \to V$. This replaces d by $d + \mathrm{ad}_\delta$.

Derivations and bimodule extensions II

Theorem

Let A be an algebra, let V and W be left A-modules.
Equip $\operatorname{Hom}(W, V)$ *with the canonical A-bimodule structure.*
Let $\operatorname{Ext}_A(W, V)$ *be the set of equivalence classes of A-module extensions $V \rightarrowtail X \twoheadrightarrow W$, where two such extensions are considered equivalent if there is a commuting diagram*

$$\begin{array}{ccccc} V & \longrightarrow & X_1 & \longrightarrow & W \\ \| & & \downarrow\cong & & \| \\ V & \longrightarrow & X_2 & \longrightarrow & W. \end{array}$$

Then $\operatorname{Ext}_A(W, V) \cong \dfrac{\operatorname{Der}(A, \operatorname{Hom}(W, V))}{\operatorname{Inn}(A, \operatorname{Hom}(W, V))}$.
An extension splits by an A-module homomorphism if and only if the corresponding derivation is inner.

Two-dimensional representations of smooth functions

- Let M be a smooth compact manifold.
- Let $\varrho\colon \mathrm{C}^\infty(M, \mathbb{C}) \to \mathbb{M}_2\mathbb{C}$ be a two-dimensional representation.
- This is reducible. So it fits into a module extension

$$V \rightarrowtail (\mathbb{C}^2, \varrho) \twoheadrightarrow W$$

  with 1-dimensional representations V and W.
- These are just characters, corresponding to points x_0, x_1 in M.
- The representation is determined by these two points and a derivation $\mathrm{C}^\infty(M) \to \mathrm{Hom}(W, V)$. Here $\mathrm{Hom}(W, V)$ is isomorphic to $\mathbb{C}$ with the bimodule structure by $f_0 \cdot \lambda \cdot f_1 := f_0(x_0)\lambda f_1(x_1)$.
- If $x_0 = x_1 = x$, the derivation is equivalent to a vector in $T_x M$.
- If $x_0 \neq x_1$, the derivation is inner.