Noncommutative Geometry IV: Differential Geometry

13. More on derivations: automorphisms and Lie algebra structure

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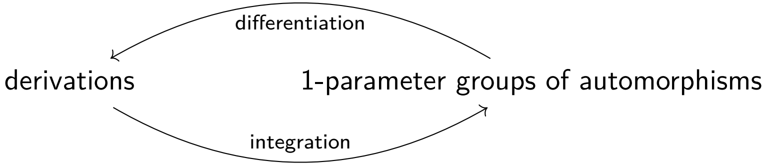
Derivations, automorphisms and Lie algebra structure

- The tangent space of a Lie group at the unit element is a Lie algebra.
- $\mathrm{Der}(A, A)$ is the tangent space of the space $\mathrm{Aut}(A)$ of algebra automorphisms $A \to A$ at the identity automorphism.
- This vague idea suggests



- $\mathrm{Der}(A, A)$ has a Lie bracket.
- We define a subgroup of inner automorphisms and show that these correspond roughly to inner derivations.
- We also discuss the physical significance of the differentiation and integration of derivations to automorphisms.

Flows on manifolds

Definition

A flow or a 1-parameter group of diffeomorphisms on M is a group homomorphism $\Phi\colon \mathbb{R} \to \mathrm{Diffeo}(M)$, $t \mapsto \Phi_t$, such that the map $\mathbb{R} \times M \to M$, $(t, m) \mapsto \Phi_t(m)$ is smooth.
The generator of the flow is the vector field

$$X\colon M \to \mathrm{T}M, \qquad X(m) := \frac{\partial}{\partial t}\Phi_t(m)\Big|_{t=0}.$$

Theorem

Let M be a smooth compact manifold and X a smooth vector field. There is a unique flow Φ with generator X.

If M is not compact, then there is still at most one flow with generator X.

Smooth 1-parameter groups on noncommutative algebras

- To talk about flows on a noncommutative algebra, we need an algebra $C^\infty(\mathbb{R}, A)$ of smooth functions $\mathbb{R} \to A$.
- This requires extra structure on A.
- We do not discuss how to define this in general.
- The definition is often clear in examples.
- For instance, $C^\infty(\mathbb{R}, C^\infty(M)) := C^\infty(\mathbb{R} \times M)$.

Definition

A smooth 1-parameter group of automorphisms of A is an algebra homomorphism $\alpha\colon A \to C^\infty(\mathbb{R}, A)$ such that the maps $\mathrm{ev}_t \circ \alpha\colon A \to A$ satisfy
$\alpha_t \circ \alpha_s = \alpha_{t+s}$ for all $s, t \in \mathbb{R}$ and $\alpha_0 = \mathrm{Id}_A$.
The generator of such a smooth 1-parameter group is the map

$$D\alpha\colon A \to A, \qquad a \mapsto \mathrm{ev}_0\left(\frac{\partial}{\partial t}\alpha(a)\right).$$

Properties of the generator

Lemma

The map $D\alpha \colon A \to A$ is a derivation.
The map α solves the differential equation $\dot{\alpha}_t = (D\alpha) \circ \alpha_t$.
The formal Taylor series $\sum_{n=0}^{\infty} \frac{\alpha^{(n)}(0)}{n!} t^n$ of α at 0 is equal to the exponential series

$$\sum_{n=0}^{\infty} \frac{t^n (D\alpha)^n(a)}{n!} =: \exp(t \cdot (D\alpha))(a).$$

Thus integrating a derivation d to a 1-parameter group of automorphisms is equivalent to defining linear operators $\exp(td) \colon A \to A$ for $t \in \mathbb{R}$ with reasonable properties such as

$$\exp(td)\exp(sd) = \exp((t+s)d), \qquad \frac{\partial}{\partial t} \exp(td) = d \exp(td).$$

Derivations need not integrate I

Example

For a smooth manifold M and $k \in \mathbb{N}$, let $\mathrm{C}^k(M)$ be the algebra of k times continuously differentiable functions on M.
Any 1-parameter group of diffeomorphisms of M generates a 1-parameter group of automorphisms of $\mathrm{C}^k(M)$.
But the latter is not smooth.
The problem is that the generating vector field of a 1-parameter group of diffeomorphisms maps $\mathrm{C}^k(M)$ only to $\mathrm{C}^{k-1}(M)$.
Thus the generator is only a derivation from $\mathrm{C}^k(M)$ to the $\mathrm{C}^k(M)$-bimodule $\mathrm{C}^{k-1}(M)$.

Derivations need not integrate II

Example

Let $A := C^\infty(\mathbb{R}, \mathbb{C})$. Define $d(f) := \mathrm{i}f'$.
On the subalgebra of holomorphic functions $\mathbb{C} \to \mathbb{C}$,
the 1-parameter automorphism group $\tau_{\mathrm{i}t} f(s) := f(s + \mathrm{i}t)$
integrates this vector field.
But this makes no sense for functions defined only on $\mathbb{R}$.
The vector field above does not integrate in any way
to a smooth 1-parameter group of automorphism of A.
Automorphisms of A all come from diffeomorphisms.
Thus any smooth 1-parameter group of automorphisms of A comes from a flow.
Then its generator is a real-valued vector field.

Inner automorphisms

Definition

Let A be a unital algebra and let $u \in A$ be invertible.
Define an associated inner automorphism

$$\mathrm{Ad}_u \colon A \to A, \qquad a \mapsto uau^{-1}.$$

Lemma

The map Ad_u is an algebra automorphism.
If A is *-algebra, then Ad_u is a *-automorphism of A if and only if u is unitary.
$\mathrm{Ad}_1 = \mathrm{Id}_A$ *and* $\mathrm{Ad}_{uv} = \mathrm{Ad}_u \circ \mathrm{Ad}_v$ *for all $u, v \in A$.*
That is, $u \mapsto \mathrm{Ad}_u$ is a group homomorphism from the group of invertible elements in A to the automorphism group of A.
And $\mathrm{Ad}_u = \mathrm{Id}_A$ if and only if u belongs to the centre of A.

- Let $A^\times$ denote the group of invertible elements in A.
- A 1-parameter group in A is a group homomorphism $u\colon \mathbb{R} \to A^\times$. It is smooth if there is an element of $U \in \mathsf{C}^\infty(\mathbb{R}, A)$ with $u(t) = \mathrm{ev}_t(U)$.
- The generator of u is the element $X := \mathrm{ev}_0 \frac{\partial}{\partial t} U \in A$.
- Then U has the formal power series expansion $\sum_{n=0}^{\infty} \frac{(tX)^n}{n!}$ at 0 and solves the differential equation $\dot{U}(t) = X \cdot U(t)$.
- This suggests to interpret $u(t)$ as $\exp(tX)$.
- Taking generators is compatible with making inner derivations and automorphisms:
$$\frac{\partial}{\partial t} \mathrm{Ad}_U = \mathrm{ad}_X.$$

Physical interpretation I

- Quantum mechanics describes a physical system by its *-algebra of observables A.
- If the time evolution does not depend explicitly on time and the system is closed, then its time evolution is a 1-parameter group of automorphisms of A.
- The time evolution is not smooth for the usual choice of A. There is a largest subalgebra A_0 of A on which it is smooth.
- The derivation of A_0 that generates the time evolution may be interpreted as the energy.
- Symmetries of the system like a translation or rotation symmetry give further 1-parameter groups of automorphisms. Again, these become smooth on a suitable dense subalgebra. The generator of the translation $t\vec{v}$ describes the $\vec{v}$-component of the momentum. The generator of rotations around an axis describes the corresponding angular momentum.

Physical interpretation I

- Are energy, momenta, and angular momenta observables of the system? That is, do they belong to the algebra A?
- Mathematically, this means that these derivations are inner.
- But these observables are usually "unbounded." The energy of a system should be bounded below, but is usually not bounded from above.
- The usual choice for A contains only bounded observables.
- Thus we would need to allow "inner" derivations and automorphisms coming from elements of a larger Lie algebra or algebra than the subalgebra of the observable algebra on which the derivation or automorphism acts.

Lie algebra structure on derivations

Definition

A Lie group is a group and a smooth manifold at the same time, such that the multiplication and inversion maps are smooth.
The tangent space $\mathfrak{g}$ of a Lie group G inherits a binary operation $[,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket.

Example

Let $G = \mathrm{Gl}(n, \mathbb{R})$ be the Lie group of all invertible $n \times n$-matrices.
Then $\mathfrak{g} = \mathbb{M}_n\mathbb{R}$ and $[X, Y]$ is the usual commutator bracket.

Definition

A Lie algebra over a field K is a K-vector space $\mathfrak{g}$ with a map
$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y) \mapsto [X, Y]$,
that is K-bilinear, anti-symmetric and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The Lie bracket on derivations

Example

Any algebra A becomes a Lie algebra for the commutator bracket $[X, Y] := XY - YX$.
In particular, $\mathrm{End}(V)$ for a vector space V is a Lie algebra.

Lemma

Define $[X, Y] := X \circ Y - Y \circ X \colon A \to A$ *for two derivations* $X, Y \in \mathrm{Der}(A, A)$.
This is again a derivation.
This bracket turns $\mathrm{Der}(A, A)$ *into a Lie algebra.*
$[\mathrm{ad}_a, \mathrm{ad}_b] = \mathrm{ad}_{[a,b]}$ *for all* $a, b \in A$.
That is, $a \mapsto \mathrm{ad}_a$ *is a Lie algebra homomorphism.*