Noncommutative Geometry IV: Differential Geometry

14. Representations and crossed products for Lie algebras

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Representations and crossed products for Lie algebras

- We define representations of Lie algebras and actions of Lie algebras on algebras by derivations.
- We define the universal enveloping algebra of a Lie algebra and a crossed product for an action of a Lie algebra on an algebra by derivations.
  This is analogous to the crossed product for group actions.
- In particular, we define a crossed product for algebra with a single derivation.
- The crossed product for the differentiation derivation on polynomials is the Weyl algebra.
- It is also generated by the canonical commutation relation, $[p, q] = \mathrm{i}\hbar$.
- We prove that the Weyl algebra is simple, describe some irreducible representations, and its derivations.

Definition

Let $\mathfrak{g}$ be a Lie algebra and V a vector space.
A representation of $\mathfrak{g}$ on V is a linear map $\varrho\colon \mathfrak{g} \to \operatorname{End}(V)$ with

$$\varrho([X, Y]) = [\varrho(X), \varrho(Y)] := \varrho(X) \cdot \varrho(Y) - \varrho(Y) \cdot \varrho(X).$$

Example

If A is an algebra, then the bracket on derivations is defined so that $\operatorname{Der}(A, A) \subseteq \operatorname{End}(A)$ is a Lie subalgebra.
Thus the canonical action of derivations on A is a representation of $\operatorname{Der}(A, A)$ on A.

Example

A continuous representation of a Lie group G on a Banach space defines a smooth representation of the Lie group on a dense subspace of “smooth vectors”.
And this gives a Lie algebra representation on this subspace.

The universal enveloping algebra

Definition

The universal enveloping algebra $U(\mathfrak{g})$ is the unital algebra generated by the set $\mathfrak{g}$ with the relations that the map $i\colon \mathfrak{g} \to U(\mathfrak{g})$ is linear and satisfies $i([X, Y]) = i(X) \cdot i(Y) - i(Y) \cdot i(X)$.

Remark

Representations of $U(\mathfrak{g})$ are equivalent to representations of $\mathfrak{g}$.

Example

Let $\mathfrak{g} = \mathbb{R}^n$ with the zero bracket.
A representation of $\mathfrak{g}$ is equivalent to
a family of n commuting operators on a vector space.
The universal enveloping algebra is isomorphic to $\mathbb{R}[x_1, \ldots, x_n]$.
$\mathbb{R}[x]$ is the universal enveloping algebra of the Lie algebra $\mathbb{R}$ with zero bracket.

Lie algebra actions on algebras

Definition

An action of a Lie algebra $\mathfrak{g}$ on an algebra A is a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Der}(A, A)$, that is, $\alpha_{[X,Y]} = [\alpha_X, \alpha_Y]$.

Definition

Let $\mathfrak{g}$ be a Lie algebra, A a unital algebra, and $\alpha\colon \mathfrak{g} \to \mathrm{Der}(A, A)$ an action of $\mathfrak{g}$ on A. A covariant representation of $(A, \mathfrak{g}, \alpha)$ on a vector space V is a pair (π, ϱ) consisting of an algebra representation $\pi\colon A \to \mathrm{End}(V)$ and a Lie algebra representation $\varrho\colon \mathfrak{g} \to \mathrm{End}(V)$, subject to the covariance condition

$$[\varrho(X), \pi(a)] = \pi(\alpha_X(a)).$$

The crossed product $A \rtimes_\alpha \mathfrak{g}$ is an algebra whose representations are equivalent to covariant representations of $(A, \mathfrak{g}, \alpha)$.

The case of a single derivation

Example

Let $\mathfrak{g}$ be $\mathbb{R}$ with zero bracket.
A Lie algebra action of $\mathbb{R}$ on an algebra A is equivalent to a single derivation d on A.
A covariant representation becomes equivalent to a pair (π, X) consisting of a representation $\pi \colon A \to \mathrm{End}(V)$ and a linear map $X \in \mathrm{End}(V)$ that satisfies $[X, \pi(a)] = \pi(d(a))$ for all $a \in A$.

Description of the crossed product

Theorem

Let $\mathfrak{g}$ be a Lie algebra and A a unital algebra.
Let $\alpha\colon \mathfrak{g} \to \mathrm{Der}(A, A)$ be an action of $\mathfrak{g}$ on A.
There is a unique associative multiplication on $A \otimes U(\mathfrak{g})$ such that

$$i_A\colon A \to A \otimes U(\mathfrak{g}), \qquad a \mapsto a \otimes 1,$$

is an algebra homomorphism,

$$i_{\mathfrak{g}}\colon \mathfrak{g} \to A \otimes U(\mathfrak{g}), \qquad X \mapsto 1 \otimes X,$$

is a Lie algebra homomorphism, and

$$[i_{\mathfrak{g}}(X), i_A(a)] = i_A(\alpha_X(a)).$$

This makes $A \otimes U(\mathfrak{g})$ the crossed product $A \rtimes_\alpha \mathfrak{g}$.

Crossed product for a single derivations

Corollary

The crossed product $A \rtimes_d \mathbb{R}$ *for a single derivation* $d \in \mathrm{Der}(A, A)$ *is* $A \otimes \mathbb{R}[t]$ *with the multiplication generated by that of* A *and the commutation relation* $[t, a] = d(a)$.

Proposition

Let $d = \mathrm{ad}_x$ *be an inner derivation on an algebra* A.
Then $A \rtimes_d \mathbb{R} \cong A \otimes \mathbb{R}[t]$.
Here the target $A \otimes \mathbb{R}[t]$ *carries the obvious multiplication* $(a \otimes t^m) \cdot (b \otimes t^n) := (ab) \otimes t^{m+n}$.

The Weyl algebra

Definition

Let A be the crossed product of $\mathbb{C}[q]$ by the derivation $d(q^n) := \mathrm{i}\hbar n q^{n-1}$ for some $\hbar \in \mathbb{C} \setminus \{0\}$.
This algebra is called Weyl algebra by mathematicians and Heisenberg algebra by mathematical physicists.

The algebra A is the universal algebra with two generators p, q that satisfy the canonical commutation relation

$$[p, q] := pq - qp = \hbar \mathrm{i}.$$

Example

Define operators p, q on $\mathbb{C}[x]$ by

$$p(f) := x \cdot f, \qquad q(f) := \frac{\hbar}{\mathrm{i}} f' = -\hbar \mathrm{i} f',$$

where f' denotes the derivative of f.
This generates a representation of the Weyl algebra.

Simplicity and uniqueness

Proposition

The Weyl algebra is simple.

Proposition

The Weyl algebra A is the unique unital algebra with two generators satisfying the canonical commutation relation.
Let B be any unital algebra and let $P, Q \in B$ satisfy $[P, Q] = \hbar \mathrm{i}$.
Then there is a unique unital algebra homomorphism $f : A \to B$ with $f(p) = P$ and $f(q) = Q$.
And f is an algebra isomorphism onto the subalgebra of B generated by P and Q.

Irreducible representations of the Weyl algebra

Proposition

Let f be a polynomial and let $V_f \subseteq \mathrm{C}^\infty(\mathbb{R})$ be the subspace of all functions of the form $g \exp(f)$ with a polynomial $g \in \mathbb{C}[x]$.
Let p and q act on V_f by $p(g \exp f) := x \cdot g \exp f$ and

$$q(g \exp f) := -\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}x}(g \exp f) = -\mathrm{i}\hbar(g' - gf') \exp f.$$

This defines an irreducible representation of A on V_f.
The irreducible representations V_{f_1} and V_{f_2} are only isomorphic if $f_2 = f_1 + c$ for a constant $c \in \mathbb{C}$.

Remark

In mathematical physics, we are mainly interested in self-adjoint solutions of the canonical commutation relation.
Any such representation of the Weyl algebra is unitarily equivalent to a direct sum of copies of the standard representation on $L^2(\mathbb{R})$.

Derivations on the Weyl algebra

Theorem

Let A be the Weyl algebra. Any derivation $A \to A$ is inner.
(*But there are non-inner derivations from the Weyl algebra into bimodules over it.*)

Lemma

The only invertible elements of the Weyl algebra A are the constant multiples of the identity, which are central.
So the only inner automorphism of A is the identity map.

Example

Translations define a 1-parameter group of automorphisms

$$\tau_t \colon A \to A, \qquad \tau_t(p^n q^m) := (p - t)^n q^m.$$

Its generator is the inner derivation $\mathrm{ad}_{q/\mathrm{i}\hbar}$.
The automorphism τ_t is not inner.
because $\exp(qt/\mathrm{i}\hbar)$ is not a polynomial in q.