Noncommutative Geometry IV: Differential Geometry

15. The universal derivation — Morita invariance

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The universal derivation — Morita invariance

- We define the bimodule of noncommutative differential forms $\Omega^1(A)$ so that
  $\{\text{derivations } A \to M\} \cong \{\text{bimodule maps } \Omega^1(A) \to M\}$.
- We define projective bimodule resolutions and use them to compute the quotient of derivations modulo inner derivations.
- We prove that this quotient
  is invariant under Morita equivalence.

Noncommutative differential 1-forms

Definition

Make $A \otimes A$ an A-bimodule in the obvious way:
$a \cdot (b \otimes c) \cdot d := (a \cdot b) \otimes (c \cdot d)$ for $a, b, c, d \in A$.
Define $\Omega^1(A) := \ker(\mathrm{mult}\colon A \otimes A \to A)$ and
$\mathrm{d}\colon A \to \Omega^1(A)$, $\mathrm{d}(a) := 1 \otimes a - a \otimes 1$.

Proposition

Let M be a unital A-bimodule and let $D\colon A \to M$ be a derivation. There is a unique bimodule map $f\colon \Omega^1(A) \to M$ with $f \circ \mathrm{d} = D$.

Corollary

The derivation $\mathrm{d}\colon A \to \Omega^1(A)$ is inner if and only if all derivations into A-bimodules are inner.

Remark

The map d is the inner derivation into $A \otimes A$ generated by $1 \otimes 1$.

Differential forms

Definition

The space $\Omega^1(M)$ of smooth 1-forms is the space of smooth sections of the cotangent bundle T^*M.

Proposition

Let V be a $\mathrm{C}^\infty(M)$-module. View it as a bimodule by $f \cdot v \cdot g = f \cdot g \cdot v$.
Let $D\colon \mathrm{C}^\infty(M) \to V$ be a derivation. There is a unique module homomorphism $\varphi\colon \Omega^1(M) \to V$ with $f \circ \mathrm{d}_{\mathrm{dR}} = D$.

Corollary

Let M be a smooth manifold and $A := \mathrm{C}^\infty(M)$. The A-module $\Omega^1(M)$ is isomorphic to the commutator quotient
$\Omega^1(A) \;/\; [A, \Omega^1(A)]$,
the quotient by the linear span of $[a, \omega]$ for $a \in A$, $\omega \in \Omega^1(A)$.

The centre using bimodule maps

Definition

Let $\mathrm{Hom}_{A,A}(V, W)$ for two A-bimodules V, W denote the space of A-bimodule homomorphisms $V \to W$.

Lemma

$\mathrm{Hom}_{A,A}(A \otimes A, M) \cong M$ for any unital A-bimodule M.

Proof.

Map $f \in \mathrm{Hom}_{A,A}(A \otimes A, M)$ to $f(1 \otimes 1)$
and $m \in M$ to the map $a \otimes b \mapsto a \cdot m \cdot b$. □

Lemma

Make A an A-bimodule by left and right multiplication.
Then $\mathrm{Hom}_{A,A}(A, M)$ is isomorphic to the centre
$Z(M) = \{m \in M : a \cdot m = m \cdot a \text{ for all } a \in A\}$.

Inner derivations using bimodule maps

Let mult: $A \otimes A \to A$ be the multiplication map. It induces a map

$$\mathrm{Hom}_{A,A}(A, M) \xrightarrow{\mathrm{mult}^*} \mathrm{Hom}_{A,A}(A \otimes A, M) \cong M, \qquad f \mapsto f \circ \mathrm{mult}.$$

Since mult is surjective, mult^* is injective.
Its image is the centre $Z(M)$ of M.

Definition

The cokernel of a map $f: X \to Y$ is the quotient $Y/f(X)$.

Since $\mathrm{ad}_m = 0$ if and only if m is central,

$$\mathrm{Inn}(A, M) \cong \frac{M}{Z(M)} = \mathrm{coker}(\mathrm{mult}^*).$$

The first two Hochschild cohomology groups

The functor $\mathrm{Hom}_{A,A}(\sqcup, M)$ maps the inclusion $\Omega^1(A) \rightarrowtail A \otimes A$ to

$$M \xrightarrow{\mathrm{ad}} \mathrm{Der}(A, M).$$

Its kernel and cokernel

$$\mathrm{HH}^0(A, M) := \ker\big(\mathrm{ad}\colon M \to \mathrm{Der}(A, M)\big) \cong Z(M),$$
$$\mathrm{HH}^1(A, M) := \mathrm{coker}\big(\mathrm{ad}\colon M \to \mathrm{Der}(A, M)\big) \cong \frac{\mathrm{Der}(A, M)}{\mathrm{Inn}(A, M)}$$

measure to what extent the restriction map
$\mathrm{Hom}_{A,A}(A \otimes A, M) \to \mathrm{Hom}_{A,A}(\Omega^1(A), M)$
is injective and surjective.

Morita invariance

Theorem

Let A and B be Morita equivalent unital algebras. Then $\mathrm{HH}^j(A, A) \cong \mathrm{HH}^j(B, B)$ for $j = 0, 1$.

Corollary

Let G be a finite group that acts freely on a smooth manifold M. Then

$$\mathrm{HH}^0(\mathrm{C}^\infty(M) \rtimes G, \mathrm{C}^\infty(M) \rtimes G) \cong \mathrm{C}^\infty(G \backslash M),$$
$$\mathrm{HH}^1(\mathrm{C}^\infty(M) \rtimes G, \mathrm{C}^\infty(M) \rtimes G) \cong \mathfrak{X}(G \backslash M),$$

vector fields on $B \backslash M$.

The Morita equivalence made explicit

- A Morita equivalence between two unital algebras A and B is given by an A, B-bimodule P and a B, A-bimodule Q and bimodule isomorphisms $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$.
- These yield an equivalence between the categories of unital A-bimodules $\mathfrak{Bimod}(A)$ and of unital B-bimodules $\mathfrak{Bimod}(B)$.
- The equivalence is $\varphi\colon \mathfrak{Bimod}(B) \to \mathfrak{Bimod}(A)$, $M \mapsto P \otimes_B M \otimes_B Q$ with inverse $M' \mapsto Q \otimes_A M' \otimes_A P$.
- $\varphi(B) := P \otimes_B B \otimes_B Q \cong P \otimes_B Q \cong A$.

Stably isomorphic maps

Definition

Two bimodule homomorphisms $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ are stably isomorphic if there are bimodules S_1 and S_2 and bimodule isomorphisms that make the following diagram commute:

$$\begin{array}{ccc} X_1 \oplus S_1 & \xrightarrow{f_1 \oplus \mathrm{Id}} & Y_1 \oplus S_1 \\ \downarrow \cong & & \downarrow \cong \\ X_2 \oplus S_2 & \xrightarrow{f_2 \oplus \mathrm{Id}} & Y_2 \oplus S_2 \end{array}$$

Remark

If f_1 and f_2 are stably isomorphic, then

$$\ker f_1 \cong \ker f_2, \qquad \operatorname{coker} f_1 \cong \operatorname{coker} f_2.$$

The idea of the proof I

Proposition

The maps $\varphi(\Omega^1 B) \to \varphi(B \otimes B)$ *and* $\Omega^1 A \to A \otimes A$ *are stably isomorphic as bimodule maps.*

- The functor $\mathrm{Hom}_{A,A}(\_, M)$ preserves stable isomorphism.
- Stably isomorphic maps have isomorphic kernels and cokernels.
- Thus the proposition implies the Morita invariance of HH^0 and HH^1.

Projective modules

Lemma

Let P be an A-bimodule. The following assertions are equivalent:

1. *the functor* $\mathrm{Hom}_{A,A}(P, \sqcup)$ *is exact, that is, if* $K \rightarrowtail E \twoheadrightarrow Q$ *is a bimodule extension, then the induced maps*

$$\mathrm{Hom}_{A,A}(P, K) \to \mathrm{Hom}_{A,A}(P, E) \to \mathrm{Hom}_{A,A}(P, Q)$$

   form an extension of Abelian groups.

2. *any surjective bimodule map* $\pi\colon E \twoheadrightarrow P$ *splits,*
   that is, there is a bimodule map $\sigma\colon P \to E$ *with* $\pi \circ \sigma = \mathrm{Id}_P$*;*
   we also call σ *a section for* π*.*

3. *if* $\pi\colon E \twoheadrightarrow Q$ *is a surjective bimodule map,*
   then any bimodule map $f\colon P \to Q$ *lifts to a bimodule map*
   $\hat{f}\colon P \to E$ *(lifting means* $\pi \circ \hat{f} = f$*).*

Definition

An A-bimodule P with these properties is called projective.

Free bimodules are projective

Example

Let V be any vector space. Turn $A \otimes V \otimes A$ into an A-bimodule by $a \cdot (b \otimes v \otimes c) \cdot d := (ab) \otimes v \otimes (cd)$.
This is called the free bimodule on V.
It is characterised by the existence of a natural isomorphism

$$\mathrm{Hom}_{A,A}(A \otimes V \otimes A, M) \cong \mathrm{Hom}(V, M)$$

for any A-bimodule M. Hence it is projective.

Schanuel's Lemma

Lemma (Schanuel's Lemma)

Let

$$K_1 \overset{i_1}{\rightarrowtail} E_1 \overset{p_1}{\twoheadrightarrow} Q \qquad \text{and} \qquad K_2 \overset{i_2}{\rightarrowtail} E_2 \overset{p_2}{\twoheadrightarrow} Q$$

be bimodule extensions with the same quotient. Assume that E_1 and E_2 are projective. Then there are bimodule isomorphisms that make the following diagram commute:

$$\begin{array}{ccccc} K_1 \oplus K_2 & \xrightarrow{i_1 \oplus \mathrm{Id}} & E_1 \oplus K_2 & \xrightarrow{(p_1, 0)} & Q \\ \downarrow \cong & & \downarrow \cong & & \| \\ K_1 \oplus K_2 & \xrightarrow{\mathrm{Id} \oplus i_2} & K_1 \oplus E_2 & \xrightarrow{(0, p_2)} & Q. \end{array}$$

Thus i_1 and i_2 are stably isomorphic.

The rest of the proof

- An equivalence of categories must preserve projective objects because they are defined in purely category theoretic terms.
- Hence both $A \otimes A$ and $\varphi(B \otimes B)$ are projective A-bimodules.
- Schanuel's Lemma shows that the maps $\Omega^1(A) \to A \otimes A$ and $\varphi(\Omega^1(B)) \to \varphi(B \otimes B)$ are stably isomorphic.
- Hence we may use $\varphi(\Omega^1(B)) \to \varphi(B \otimes B)$ to compute $\mathrm{HH}^0(A, M)$ and $\mathrm{HH}^1(A, M)$.
- Since φ is an equivalence of categories, this gives

$$\mathrm{HH}^0(A, M) \cong \mathrm{HH}^0(B, \varphi^{-1}(M)),$$
$$\mathrm{HH}^1(A, M) \cong \mathrm{HH}^1(B, \varphi^{-1}(M)).$$

- Finally, $\varphi^{-1}(B) \cong A$.