Noncommutative Geometry IV: Differential Geometry

16. From deformation quantisations to Hochschild cohomology

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From deformation quantisations to Hochschild cohomology

- We have introduced $\mathrm{HH}^0(A, M)$ and $\mathrm{HH}^1(A, M)$ last time. Here A is an algebra, M an A-bimodule.
- We will define $\mathrm{HH}^n(A, M)$ for all $n \geq 0$.
- $\mathrm{HH}^2(A, M)$ is related to certain algebra extensions.
- These are related to formal deformation quantisations.
- We will define polynomial and formal deformation quantisations and sketch the physical intuition behind.
- Then we "truncate" deformation quantisations.
- The truncated deformation quantisations are classified by $\mathrm{HH}^2(A, A)$.

Definition

A polynomial deformation quantisation of an algebra A_0 is an associative multiplication on $A_0[\hbar] := A_0 \otimes \mathbb{C}[\hbar]$ that is $\hbar$-bilinear and agrees with the multiplication in A_0 up to terms divisible by $\hbar$.

- Then $m(a,b) = \sum_{k=0}^{\infty} m_k(a,b) \cdot \hbar^k$ with $m_k \colon A_0 \times A_0 \to A_0$.
- Associativity means that

$$\sum_{k=0}^{l} m_k(a, m_{l-k}(b,c)) = \sum_{k=0}^{l} m_k(m_{l-k}(a,b), c)$$

- For $l = 1$, this says

$$a \cdot m_1(b,c) + m_1(a, b \cdot c) = m_1(a,b) \cdot c + m_1(a \cdot b, c).$$

- These equations are non-linear in (m_k) for $l \geq 2$.

Formal deformation quantisation

In a polynomial deformation quantisation, the series $\sum_{l=0}^{\infty} m_l(a,b)\hbar^l$ must be finite for each fixed a, b. This is hard to control.

Definition

Let A_0 be an algebra and let $A_0[\![\hbar]\!]$ be the $\mathbb{C}[\![\hbar]\!]$-module of formal power series in one variable with coefficients in A_0.
A formal deformation quantisation of A_0 is an associative multiplication on $A := A_0[\![\hbar]\!]$ that is $\hbar$-bilinear and agrees with the multiplication in A_0 up to terms divisible by $\hbar$.

An example of a formal deformation quantisation

Example

- For $\hbar \in \mathbb{R}$, let $\alpha_\hbar \colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{C}[t, t^{-1}])$ be the automorphism that is induced by rotation with angle $\hbar$:
- $\alpha_\hbar(t^n) := \exp(2\pi i \hbar n) t^n$ for all $n \in \mathbb{Z}$.
- Let $B := \mathbb{C}[t, t^{-1}][\![\hbar]\!]$ and define an $\hbar$-linear automorphism $\alpha \in \mathrm{Aut}(B)$ by

$$\alpha(t^n \hbar^m) = t^n \exp(2\pi i \hbar n) \hbar^m := t^n \cdot \sum_{j=0}^{\infty} \frac{(2\pi i n)^j}{j!} \hbar^{m+j}.$$

- The crossed product $B \rtimes_\alpha \mathbb{Z}$ is a formal deformation quantisation of $\mathbb{C}[t, t^{-1}] \rtimes_0 \mathbb{Z}$. This deformation quantisation is not polynomial because it involves exponential functions.

Equivalence of formal deformation quantisations

Definition

Two formal deformation quantisations m and m' are equivalent if there is an invertible $\hbar$-linear map $\Psi\colon A_0[\![\hbar]\!] \to A_0[\![\hbar]\!]$ with $m(\Psi(a), \Psi(b)) = \Psi(m'(a, b))$ for all $a, b \in A_0[\![\hbar]\!]$ and $\Psi(a) \equiv a \bmod (\hbar)$ for all $a \in A_0$.

- Compare coefficients: (m_l) and (m'_l) are equivalent iff

$$\sum_{j=0}^{l} \sum_{k=0}^{l-j} m_{l-j-k}(\Psi_j(a), \Psi_k(b)) = \sum_{j=0}^{l} \Psi_j(m'_{l-j}(a, b))$$

  for all $l \in \mathbb{N}$, $a, b \in A_0$.
- For $l = 1$, it says that

$$m_1(a, b) + \Psi_1(a) \cdot b + a \cdot \Psi_1(b) = m'_1(a, b) + \Psi_1(a \cdot b).$$

- Any family of linear maps $\Psi_l\colon A_0 \to A_0$ with $\Psi_0 = \mathrm{Id}_A$ gives an invertible $\hbar$-linear map Ψ on $A_0[\![\hbar]\!]$. And then $m'(a, b) := \Psi^{-1}\big(m(\Psi(a), \Psi(b))\big)$ is a formal deformation quantisation. It is equivalent to m by construction.

Definition

- Let A be an algebra and let M be an A-bimodule. An M-valued Hochschild 2-cocycle on A is a bilinear map $\omega\colon A \times A \to M$ satisfying

$$a \cdot \omega(b, c) + \omega(a, b \cdot c) = \omega(a, b) \cdot c + \omega(a \cdot b, c).$$

- A bilinear map $\omega\colon A \times A \to M$ is called a Hochschild 2-coboundary if it is of the form

$$\partial\psi(a, b) := a \cdot \psi(b) + \psi(a) \cdot b - \psi(a \cdot b)$$

for some linear map $\psi\colon A \to M$.

- Any Hochschild 2-coboundary is a Hochschild 2-cocycle.
- Hence we may define

$$\mathrm{HH}^2(A, M) := \frac{\{\text{Hochschild 2-cocycles}\}}{\{\text{Hochschild 2-coboundaries}\}}.$$

Hochschild 2-cocycles and deformation quantisation

Proposition

There is a bijection between $\mathrm{HH}^2(A, A)$ *and equivalence classes of associative multiplications on* $A[\![\hbar]\!]/(\hbar^2) \cong A \oplus \hbar \cdot A$
that specialise to the given product for $\hbar = 0$.

Definition

An algebra extension $I \rightarrowtail E \twoheadrightarrow Q$ is called a
square-zero extension of Q by I if $i_1 \cdot i_2 = 0$ for all $i_1, i_2 \in I$.

Theorem

Let A *be an algebra and let* M *be an* A*-bimodule.*
There is a natural bijection between $\mathrm{HH}^2(A, M)$ *and equivalence classes of square-zero extensions of* A *by* M.
Two square-zero extensions of A *by* M *are equivalent if there is a commuting diagram*

$$\begin{array}{ccccc} M & \longrightarrow & E_1 & \longrightarrow & A \\ \| & & \downarrow \cong & & \| \\ M & \longrightarrow & E_2 & \longrightarrow & A. \end{array}$$

The case of trivial second Hochschild cohomology

Theorem

Let A be an algebra with $\mathrm{HH}^2(A, A) = 0$.
Then all formal deformation quantisations are trivial.

Higher Hochschild cohomology

Definition

Let A be an algebra and let M be an A-bimodule.

- The Hochschild coboundary of an n-linear map $\varphi\colon A^n \to M$ is $\partial\varphi\colon A^{n+1} \to M$,

$$\begin{gathered}\partial\varphi(a_0, \ldots, a_n) := a_0 \cdot \varphi(a_1, \ldots, a_n) - \varphi(a_0 \cdot a_1, a_2, \ldots, a_n) \\ +\varphi(a_0, a_1 \cdot a_2, a_3, \ldots, a_n) - \varphi(a_0, a_1, a_2 \cdot a_3, a_4, \ldots, a_n) \\ \pm \cdots + (-1)^n \varphi(a_0, \ldots, a_{n-2}, a_{n-1} \cdot a_n) \\ +(-1)^{n+1}\varphi(a_0, \ldots, a_{n-2}, a_{n-1}) \cdot a_n.\end{gathered}$$

- We call ψ a Hochschild cocycle if $\partial\psi = 0$.
- The Hochschild cohomology is

$$\mathrm{HH}^n(A, M) := \frac{\{\text{Hochschild } n\text{-cocycles}\}}{\{\text{Hochschild } n\text{-coboundaries}\}}.$$

Normalised Hochschild cocycles

Definition

Let A be a unital algebra and let M be a unital A-bimodule.
A Hochschild n-cochain $\omega\colon A^n \to M$ is called normalised if $\omega(a_1, \dots, a_n)$ vanishes whenever $a_i = 1$ for some $i \in \{1, \dots, n\}$.

Lemma

Let A be a unital algebra, M a unital A-bimodule, and $n \in \mathbb{N}$.
Any Hochschild cocycle $\omega\colon A^n \to M$ is cohomologous to a normalised one.
If two normalised Hochschild cocycles $\omega_1, \omega_2\colon A^n \rightrightarrows M$ are cohomologous, then there is a normalised Hochschild cochain $\psi\colon A^{n-1} \to M$ with $\partial\psi = \omega_1 - \omega_2$.
The Hochschild coboundary of a normalised Hochschild cochain is again normalised.