Noncommutative Geometry IV: Differential Geometry

17. Computing Hochschild cohomology with projective resolutions

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Computing Hochschild cohomology with projective resolutions

- We are going to use more and more homological algebra.
- This needs a lot of new language.
- Today we introduce projective resolutions
  and use them to compute Hochschild cohomology.
- This uses two ideas.
  - Hochschild cohomology can be obtained from the
    bar resolution, a certain "projective bimodule resolution".
  - All projective bimodule resolutions of a module are
    "chain homotopy equivalent".

The language of chain complexes

- Chain complex $C_\bullet$ of A-modules: sequence of A-modules $(C_n)_{n\in\mathbb{Z}}$ with boundary maps $d_n\colon C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$.
- An n-cycle in $C_\bullet$ is $x \in C_n$ with $d_n(x) = 0$.
- An n-boundary in $C_\bullet$ is $d_{n+1}(x)$ with $x \in C_{n+1}$.
- cycles are homologous if they differ by a boundary.
- The nth homology of a chain complex is the quotient of n-cycles by n-boundaries, $\mathrm{H}_n(C_\bullet) := \ker(d_n) \,/\, d_{n+1}(C_{n+1})$.
- A chain complex is called exact if all its cycles are boundaries or, equivalently, $\mathrm{H}_n(C_\bullet) = 0$ for all $n \in \mathbb{Z}$.
- A cochain complex $C^\bullet$ is similar, but has coboundary maps $d^n\colon C^n \to C^{n+1}$. The cochain complex analogues of cycles, boundaries, and homology for cochain complexes are cocycles, coboundaries and cohomology.
  The nth cohomology is written $\mathrm{H}^n(C_\bullet)$.

The Hochschild complex

Example

The Hochschild cochains $A^\bullet \to M$ with the Hochschild coboundary form a cochain complex (of vector spaces).
Its cohomology is the Hochschild cohomology $\mathrm{HH}^n(A, M)$.

Projective resolutions

Definition

A projective resolution of an A-bimodule M is a chain complex $P_\bullet$ of projective A-bimodules with $P_n = 0$ for $n < 0$ and with an augmentation map $d_0 \colon P_0 \to M$ with $d_0 \circ d_1 = 0$, such that the following augmented chain complex is exact:

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0 \to 0 \to \cdots$$

Theorem

Let A be a unital algebra and let M be an A-bimodule. Let $P_\bullet \to A$ be a projective resolution of A by unital A-bimodules, with boundary maps d_n^P. Let $\mathrm{Hom}_{A,A}(P_\bullet, M)$ be the cochain complex with A-bimodule maps $f : P_n \to M$ as n-cochains and the coboundary map $d^n(f) := (-1)^{n+1} f \circ d_{n+1}^P$. Then

$$\mathrm{HH}^n(A, M) \cong \mathrm{H}^n\big(\mathrm{Hom}_{A,A}(P_\bullet, M)\big).$$

The bar resolution

Definition

- Let $\bar{A} := A/\mathbb{C}\cdot 1$ and let $\mathrm{Bar}_n(A) := A \otimes \bar{A}^{\otimes n} \otimes A$ for $n \geq 0$; here it is understood that $\mathrm{Bar}_0(A) := A \otimes A$.
- Define $b' \colon \mathrm{Bar}_n(A) \to \mathrm{Bar}_{n-1}(A)$ by letting $b'(a_0 \otimes \cdots \otimes a_{n+1})$ be

$$\sum_{j=0}^{n} (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes a_j \cdot a_{j+1} \otimes a_{j+2} \otimes \cdots \otimes a_{n+1}.$$

Lemma

The map b' is well defined and $b' \circ b' = 0$.

Proposition

$\mathrm{Hom}_{A,A}(\mathrm{Bar}_\bullet(A), M)$ *is naturally isomorphic to the normalised Hochschild cochain complex for A with coefficients in M.*

Chain maps

Definition

A chain map between two chain complexes of A-modules $C_\bullet$ and $D_\bullet$ is a sequence of A-module maps $f_n \colon C_n \to D_n$ with $f_n \circ d^C = d^D \circ f_{n+1}$ for all $n \in \mathbb{Z}$.
That is, the following diagram commutes:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_{n+2} & \xrightarrow{d^C_{n+2}} & C_{n+1} & \xrightarrow{d^C_{n+1}} & C_n & \xrightarrow{d^C_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+2}} & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & D_{n+2} & \xrightarrow[d^D_{n+2}]{} & D_{n+1} & \xrightarrow[d^D_{n+1}]{} & D_n & \xrightarrow[d^D_n]{} & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

Chain homotopies

Definition

Let $C_\bullet$ and $D_\bullet$ be chain complexes.
Let $\mathrm{Hom}(C_\bullet, D_\bullet)$ be the chain complex whose k-chains for $k \in \mathbb{Z}$ are arbitrary sequences of maps $f_n \colon C_n \to D_{n+k}$ and whose boundary map maps (f_n) to the sequence of maps

$$d^D_{n+k} \circ f_n - (-1)^k f_{n-1} \circ d^C_n \colon C_n \to D_{n+k-1}.$$

Definition

Two chain maps $f_\bullet, g_\bullet \colon C_\bullet \rightrightarrows D_\bullet$ are chain homotopic if they are homologous in $\mathrm{Hom}(C_\bullet, D_\bullet)$, that is, there is a sequence of maps $s_n \colon C_n \to D_{n+1}$ with $d^D_{n+1} \circ s_n + s_{n-1} \circ d^C_n = f_n - g_n$. This sequence (s_n) is also called a chain homotopy between $f_\bullet$ and $g_\bullet$.
A chain map is called null homotopic if it is homotopic to the zero map. A chain complex is called contractible if its identity map is homotopic to the zero map. A 1-chain s of $\mathrm{Hom}(C_\bullet, C_\bullet)$ with $d(s) = \mathrm{Id}_{C_\bullet}$ is called a contracting homotopy of $C_\bullet$.

Chain homotopy equivalence

Lemma

Let $f, g\colon C_\bullet \rightrightarrows D_\bullet$ *be two homotopic chain maps. Then* f *and* g *induce the same map on homology.*

Definition

A chain map $f_\bullet\colon C_\bullet \to D_\bullet$ is called a (chain) homotopy equivalence if there is a chain map $g\colon D_\bullet \to C_\bullet$ such that $f_\bullet \circ g_\bullet$ and $g_\bullet \circ f_\bullet$ are homotopic to the identity maps on $D_\bullet$ and $C_\bullet$.
Two chain complexes are called (chain) homotopy equivalent if such a chain homotopy exists.

Proposition

Let $\varphi_\bullet\colon C_\bullet \to D_\bullet$ *be a homotopy equivalence between two chain complexes of A-bimodules and let* F *be an additive functor on the category of A-bimodules. Then* $F(\varphi_\bullet)$ *induces an isomorphism on homology,* $\mathrm{H}_n(F(C_\bullet)) \to \mathrm{H}_n(F(D_\bullet))$.

The uniqueness of projective resolutions

Theorem

Let M and M' be two A-bimodules and let $P_\bullet \to M$ and $P'_\bullet \to M'$ be projective A-bimodule resolutions.
Then any bimodule homomorphism $f : M \to M'$ lifts to a bimodule homomorphism chain map $P_\bullet \to P'_\bullet$; this lifting is unique up to chain homotopy. In symbols:

$$\mathrm{H}_0(\mathrm{Hom}_{A,A}(P_\bullet, P'_\bullet)) \cong \mathrm{Hom}_{A,A}(M, M').$$

Corollary

All projective A-bimodule resolutions of an A-bimodule M are homotopy equivalent as chain complexes of A-bimodules.

Application to Hochschild cohomology

Theorem

Let A be a unital algebra and let M be an A-bimodule. Let $P_\bullet \to A$ be a projective resolution of A by unital A-bimodules, with boundary maps d_n^P. Then

$$\mathrm{HH}^n(A, M) \cong \mathrm{H}^n\big(\mathrm{Hom}_{A,A}(P_\bullet, M)\big).$$

Proof.

- If $P_\bullet$ is the bar resolution, then $\mathrm{Hom}_{A,A}(P_\bullet, M)$ is the normalised Hochschild complex.
- If $P_\bullet$ is any projective bimodule resolution, then $P_\bullet$ is chain homotopy equivalent to the bar resolution as a complex of bimodules.
- This implies that an isomorphism on cohomology. □

Computation for semisimple algebras

- Let A be a semi-simple, finite-dimensional algebra.
- Then A itself is projective as an A-bimodule.
- This gives a very short projective bimodule resolution with $P_n = 0$ for $n \geq 1$ and $P_0 = A$.
- $\mathrm{HH}^n(A, M)$ is the cohomology of the chain complex

$$\cdots 0 \to 0 \to 0 \to \mathrm{Hom}_{A,A}(A, M) \to 0 \to 0 \to \cdots .$$

- That is, $\mathrm{HH}^n(A, M) = 0$ for $n \geq 1$ and $\mathrm{HH}^0(A, M) \cong \mathrm{Hom}_{A,A}(A, M)$, the centre of M.