Noncommutative Geometry IV: Differential Geometry

18. Hochschild cohomology of algebras of polynomials

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Hochschild cohomology of algebras of polynomials

- We are going to compute the Hochschild cohomology of $\mathbb{C}[x_1, \ldots, x_n]$.
- We use a small projective bimodule resolution.
- We use the same resolution to define the Taylor spectrum of an n-tuple of commuting operators on a vector space.
- In the next lecture, we will use a similar projective bimodule resolution for the algebra of smooth functions on $\mathbb{R}^n$ and then on any manifold.
- The Hochschild–Kostant–Rosenberg Theorem computes the Hochschild cohomology for the commutative algebras of polynomial functions on all smooth, affine varieties over fields of characteristic 0.
  It needs more algebraic geometry techniques, which I do not want to develop in this course.

Koszul's chain complex

V a finite-dimensional $\mathbb{C}$-vector space

$\Lambda^k V$ its kth exterior power

V^* the dual space $\mathrm{Hom}(V, \mathbb{C})$ of V

η element of V^*

$i(\eta)$ boundary map $\Lambda^k V \to \Lambda^{k-1} V$, $v_1 \wedge \cdots \wedge v_k \mapsto \sum_{j=1}^{k} (-1)^{j-1} \eta(v_j) v_1 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_k$.

- For $k = 1$, $i(\eta)\colon V \cong \Lambda^1 V \to \Lambda^0 V \cong \mathbb{C}$ maps $v \mapsto \eta(v)$.
- The antisymmetry of the $\wedge$-product implies $i(\eta) \circ i(\eta) = 0$.
- So $0 \to \Lambda^n V \xrightarrow{i(\eta)} \Lambda^{n-1} V \to \cdots \to \Lambda^1 V \xrightarrow{i(\eta)} \Lambda^0 V \to 0$ is a chain complex – the Koszul complex for V and η.

Koszul's resolution

Lemma

If $\eta = 0$*, then* $i(\eta) = 0$ *as well.*
Otherwise, let $v \in V$ *satisfy* $\eta(v) = 1$*. Then the maps*

$$\lambda_v \colon \Lambda^k V \to \Lambda^{k+1} V, \qquad \omega \mapsto v \wedge \omega,$$

form a contracting homotopy of the Koszul complex, that is,
$i(\eta) \circ \lambda_v + \lambda_v \circ i(\eta) = \mathrm{Id}$*.*
So the Koszul complex is exact.

A projective resolution for the polynomial algebra

A is $\mathbb{C}[\eta_1, \ldots, \eta_n]$

V is $\mathbb{C}^n$

C_k is $A \otimes \Lambda^k V = \mathbb{C}[\eta_1, \ldots, \eta_n] \otimes \Lambda^k \mathbb{C}^n$

d_k map $C_k \to C_{k-1}$, $d_k(f \otimes \omega)(\eta) := f(\eta) i(\eta)(\omega)$.
if $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$, then

$$d_k(f \otimes \omega) = \sum_{j=1}^{n} f \cdot \eta_j \otimes i(e_j)(\omega).$$

d_0 augmentation map $A \cong C_0 \to \mathbb{C}$, $f \mapsto f(0)$.
Clearly, $d_1 \circ d_0 = 0$.

A projective resolution for the polynomial algebra

Proposition

The chain complex $(C_\bullet, d_\bullet)$ *is a free* A*-module resolution of* $\mathbb{C}$ *with the* A*-module structure* $f \cdot x := f(0) \cdot x$ *for all* $f \in A$, $x \in \mathbb{C}$.

Theorem

Let $A := \mathbb{C}[x_1, \ldots, x_n]$. *Let* $C_k := A \otimes A \otimes \Lambda^k \mathbb{C}^n$ *with the usual free bimodule structure and define*

$$d_k \colon A \otimes A \otimes \Lambda^k \mathbb{C}^n \to A \otimes A \otimes \Lambda^{k-1} \mathbb{C}^n,$$
$$d_k(f \otimes \omega)(x, y) := f(x, y) \otimes (x - y) \wedge \omega$$

for all $f \in A \otimes A$, $\omega \in \Lambda^k \mathbb{C}^n$, $x, y \in \mathbb{C}^n$.
With the augmentation map $d_0(f)(x) := f(x, x)$, *this is a free* A*-bimodule resolution of* A.

A useful exactness criterion

Lemma

Let $C_\bullet$ *be a chain complex and let* $C_\bullet^{(k)}$ *be an increasing filtration by subcomplexes, that is,* $\partial(C_\bullet^{(k)}) \subseteq C_\bullet^{(k)}$ *for all* $k \in \mathbb{N}$*.*
Assume that $\bigcup C_\bullet^{(k)} = C_\bullet$ *and* $C_\bullet^{(0)} = \{0\}$*.*
If the quotient complexes $C_\bullet^{(k+1)} \,/\, C_\bullet^{(k)}$ *are exact for all* $k \in \mathbb{N}$*, then* $C_\bullet$ *is exact.*

The Taylor spectrum

- We are going to define a joint spectrum for several commuting operators on a vector space.
- We interpret the operators as a module over $\mathbb{C}[x_1, \ldots, x_n]$.
- The spectrum of a single linear operator T on a $\mathbb{C}$-vector space V consists of all $\lambda \in \mathbb{C}$ for which

$$0 \to V \xrightarrow{T-\lambda} V \to 0$$

  is not exact.
- $T - \lambda$ generates a $\mathbb{C}[t]$-module structure on V. The cochain complex above is $\mathrm{Hom}_{\mathbb{C}[t]}(P_\bullet, V)$ for the Koszul resolution

$$P_\bullet := (0 \to \mathbb{C}[t] \xrightarrow{t} \mathbb{C}[t] \xrightarrow{\mathrm{ev}_0} \mathbb{C}).$$

  Thus the spectrum of T is related to a projective $\mathbb{C}[t]$-module resolution of $\mathbb{C}$.

The Taylor spectrum

- Let $A := \mathbb{C}[t_1, \ldots, t_n]$.
- An n-tuple of commuting linear operators on V is equivalent to an A-module structure on V, where T_j is the operator of multiplication by t_j.
- Given $\lambda \in \mathbb{C}^n$, give V the A-module structure for the n-tuple $T - \lambda = (T_j - \lambda_j)$.
- Let (C_k, d_k) be the free A-module resolution of $\mathbb{C}$ above, $C_k := A \otimes \Lambda^k \mathbb{C}^n$.
- Then $\mathrm{Hom}_A(C_k, V)$ is the cochain complex with entries $L_k := \Lambda^k \mathbb{C}^n \otimes V$ and the boundary map

$$d_k(e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes v)$$

$$:= \sum_{j=1}^{k} (-1)^{j-1} e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_k \otimes (T_{i_j} - \lambda_{i_j})(v).$$

This chain complex is also called a Koszul complex.

The Taylor spectrum and some of its properties

Definition

The Taylor spectrum of an n-tuple of commuting operators is the set of all $\lambda \in \mathbb{C}^n$ for which the Koszul complex above is not exact.

Proposition

Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators on a vector space V. Let $W \subseteq V$ be T_j-invariant for $j = 1, \ldots, n$. Then T induces n-tuples $T|_W$ and $T|_{V/W}$ of commuting operators on W and V/W. The Taylor spectrum of T is contained in the union of the Taylor spectra of $T|_W$ and $T|_{V/W}$.

Proposition

If V is finite-dimensional,
then the Taylor spectrum of T is the set of joint eigenvalues of T.

Lemma

If $T_j - \lambda_j$ is invertible for some j,
then λ is not in the Taylor spectrum of T.