Noncommutative Geometry IV: Differential Geometry

19. Hochschild cohomology for algebras of smooth functions

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Hochschild cohomology for algebras of smooth functions

- We identify the continuous Hochschild cohomology of $C^\infty(X)$ for a smooth manifold X with the spaces of k-vector fields on X for $k \geq 0$.
- To prove the result above, we change the definitions of "projective" and "resolution", taking into account topologies on modules.
- The "projective resolutions" in the new sense are very similar to the Koszul resolutions for polynomials.

The main theorem

Definition

A *k-vector field* is a section of the kth exterior power of the tangent bundle of X. That is, $\pi(x) \in \Lambda^k \mathsf{T}_x X$ for all $x \in X$.

Theorem

The kth continuous Hochschild cohomology of $\mathsf{C}^\infty(X)$ with coefficients in $\mathsf{C}^\infty(X)$ is naturally isomorphic to the space $\mathfrak{X}^k(X)$ of smooth k-vector fields on X. The isomorphism maps a k-vector field π to the Hochschild k-cocycle

$$\Sigma_\pi \colon \mathsf{C}^\infty(X)^k \to \mathsf{C}^\infty(X),$$
$$\Sigma_\pi(f_1, \dots, f_k)(x) := \langle \pi(x) \,|\, Df_1(x), \cdots, Df_k(x) \rangle.$$

Here $Df_j(x) \in (\mathsf{T}_x X)^*$ is the derivative of f_j at x.

Why change the notion of bimodule?

- We want to compute the continuous Hochschild cohomology using "projective resolutions".
- We need topological modules to talk about continuous maps.
- The algebraic tensor product $\mathsf{C}^\infty(\mathbb{R}^n) \otimes \mathsf{C}^\infty(\mathbb{R}^n)$ is not a nice space of functions.
- We want to complete it to $\mathsf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.
- But $\mathsf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is not projective because the diagonal restriction map $\mathsf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n) \to \mathsf{C}^\infty(\mathbb{R}^n)$ does not lift to a bimodule map $\mathsf{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n) \to \mathsf{C}^\infty(\mathbb{R}^n) \otimes \mathsf{C}^\infty(\mathbb{R}^n)$.
- We restrict to complete topological modules and bimodules.

The right bimodules over $\mathrm{C}^\infty(X)$

Theorem

Let M and N be smooth manifolds and V a complete locally convex topological vector space. A continuous bilinear map $b\colon \mathrm{C}^\infty(M) \times \mathrm{C}^\infty(N) \to V$ extends uniquely to a continuous linear map $l\colon \mathrm{C}^\infty(M \times N) \to V$. The complete projective topological tensor product of $\mathrm{C}^\infty(M)$ and $\mathrm{C}^\infty(N)$ is naturally isomorphic to $\mathrm{C}^\infty(M \times N)$.

Theorem

A complete locally convex topological bimodule over $\mathrm{C}^\infty(X)$ is the same as a complete locally convex topological module over $\mathrm{C}^\infty(X \times X)$.

Why change the definition of exactness?

Lemma

If V is a finite-dimensional vector space, then $\mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n, V)$ is a projective $\mathrm{C}^\infty(\mathbb{R}^n)$-bimodule.

Proof.

$\mathrm{Hom}_{\mathrm{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n)}(\mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n, V), M) \cong \mathrm{Hom}(V, M).$ □

- The bar resolution uses $\mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n, V)$ for infinite-dimensional V, say, $V = \mathrm{C}^\infty(\mathbb{R}^n \times \cdots \times \mathbb{R}^n)$.
- This is only projective for extensions of topological modules with a continuous linear section.
- If a chain complex $C_\bullet$ has a continuous contracting homotopy, then the chain complex $\mathrm{Hom}(V, C_\bullet)$ of continuous linear maps $V \to C_\bullet$ is again contractible, hence exact.
- We only allow exact chain complexes and resolutions with a continuous contracting homotopy and call these admissible.

Continuous Hochschild cohomology and projective resolutions

Theorem

Let A be a complete, locally convex topological unital algebra and let M be a complete, locally convex topological unital A-module. Let $P_\bullet$ be a chain complex with an augmentation $P_0 \to A$. Suppose that the bimodules P_n are relatively projective and that the augmented chain complex is admissibly exact. Then $\mathrm{HH}^n_{\mathrm{cont}}(A, M) \cong \mathrm{H}^n(\mathrm{Hom}_{A,A}(P_\bullet, M))$.

- Replacing $\mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}[y_1, \ldots, y_n]$ in the Koszul resolution for polynomials by $\mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ gives a projective resolution as above.
- This allows to compute the continuous Hochschild cohomology of $\mathrm{C}^\infty(\mathbb{R}^n)$.
- The computation for a manifold X uses the Koszul resolution for the tangent bundle $\mathrm{T}X$ and another trick to replace $X \times X$ by a tubular neighbourhood $\mathrm{T}X$ around the diagonal.