Noncommutative Geometry IV: Differential Geometry

20. Quasi-free algebras and their Hochschild cohomology

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Quasi-free algebras and their Hochschild cohomology

Question

What can we say about an algebra A
when A has a "short" projective A-bimodule resolution?

- Today we consider the case
  when A has a projective bimodule resolution of length 1.
- Equivalent characterisations:
  - any square-zero algebra extension of A splits;
  - any nilpotent extension of A splits;
  - $\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M;
  - $\mathrm{HH}^k(A, M) = 0$ for all $k \geq 2$ and all A-bimodules M;
  - $\Omega^1(A)$ is a projective A-bimodule.
- We give many examples of quasi-free algebras:
  $\mathbb{C}$, $\mathbb{C}[t]$, $\mathbb{C}[t, t^{-1}]$; group algebra of the dihedral group;
  Toeplitz algebra; quiver algebras and Leavitt path algebras.
- Hochschild cohomology for quasi-free algebras reduces to HH^0 and HH^1.

The definition and a first consequence

Definition

An algebra is called quasi-free if
any square-zero algebra extension $I \rightarrowtail E \twoheadrightarrow A$ splits
by an algebra homomorphism $A \to E$.
This is equivalent to $\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M.

Lemma

A unital algebra A is quasi-free if and only if
$\mathrm{HH}^2(A, M) = 0$ *for all unital A-bimodules,*
if and only if any square-zero extension $I \rightarrowtail E \twoheadrightarrow A$ with unital E splits by a unital algebra homomorphism.

Theorem

If A is quasi-free, then any formal deformation quantisation of A is equivalent to the trivial one: $m(a, b) = a \cdot b$ for all $a, b \in A$.

Examples of quasi-free algebras 1

Proposition

The field $\mathbb{C}$ is quasi-free.
If $I \rightarrowtail E \twoheadrightarrow A$ is a square-zero extension,
then any idempotent $p \in A$ lifts to an idempotent $\hat{p} \in E$.

Proposition

The group algebra of the infinite dihedral group D_∞ is quasi-free.

Example

The polynomial algebra $\mathbb{C}[p]$ is quasi-free, even free:
any algebra extension $I \rightarrowtail E \twoheadrightarrow \mathbb{C}[p]$ with unital E splits
by a unital algebra homomorphism $\mathbb{C}[p] \to E$:
lift $p \in \mathbb{C}[p]$ to some $e \in E$ and map $p^n \mapsto e^n$ for $n \in \mathbb{N}$.

Example

The polynomial algebra $\mathbb{C}[p, q]$ is not quasi-free because the Weyl algebra deformation provides a non-split square-zero extension.

Examples of quasi-free algebras 2

Proposition

Let Q be a quiver with countably many vertices.
Its quiver algebra is quasi-free.

Proposition

The algebra $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials is quasi-free.

Proposition

The Toeplitz algebra is quasi-free.

Theorem

Let Γ be a directed graph with countably many vertices and let $\Gamma_0' \subseteq \Gamma_0$ be a set of regular vertices.
Then the relative Leavitt path algebra $L(\Gamma, \Gamma_0')$ is quasi-free.

Nilpotent extensions of quasi-free algebras split

Definition

An algebra extension $I \rightarrowtail E \twoheadrightarrow Q$ is called nilpotent if there is $k \in \mathbb{N}$ with $I^k = 0$.

Theorem

Let A be a unital algebra. The following are equivalent:

- *any square-zero extension $I \rightarrowtail E \twoheadrightarrow A$ splits;*
- *for any square-zero extension $I \rightarrowtail E \twoheadrightarrow Q$, any algebra homomorphism $A \to Q$ lifts to an algebra homomorphism $A \to E$;*
- *any nilpotent extension $I \rightarrowtail E \twoheadrightarrow A$ splits;*
- *for any nilpotent extension $I \rightarrowtail E \twoheadrightarrow Q$, any algebra homomorphism $A \to Q$ lifts to an algebra homomorphism $A \to E$.*

More equivalent characterisations of quasi-freeness

Theorem

Let A be a unital algebra. The following are equivalent:

- $\Omega^1(A)$ *is a projective A-bimodule;*
- *A has a projective A-bimodule resolution of length* 1*;*
- *A is quasi-free, that is,* $\mathrm{HH}^2(A, M) = 0$ *for all A-bimodules M.*