Noncommutative Geometry IV: Differential Geometry

21. Hochschild cohomology of the Weyl algebra — De Rham cohomology

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Summer Term 2020

Part 1: Hochschild cohomology of the Weyl algebra

- The Koszul resolution for two variables also gives a projective bimodule resolution for the Weyl algebra.
- We use this to compute the Hochschild cohomology of the Weyl algebra.
- It turns out that the Hochschild cohomology for $\hbar = 0$ and $\hbar \neq 0$ are quite different.
- Hochschild cohomology is
  not invariant under deformation quantisation.

The Koszul resolution once again

- Fix $\hbar \in \mathbb{C}$.
- Let $A = \mathbb{C}\langle p, q \mid [p, q] = \mathrm{i}\hbar \rangle$.
- Let $P_k := A \otimes \Lambda^k(\mathbb{C}^2) \otimes A$.
  That is, $P_0 \cong P_2 \cong A \otimes A$, $P_1 \cong (A \otimes A) \oplus (A \otimes A)$.
- Define $d_k \colon P_k \to P_{k-1}$ for $k = 2, 1$ and $d_0 \colon A \otimes A \to A$:

$$
\begin{aligned}
d_2(a \otimes e_1 \wedge e_2 \otimes b) &:= a \cdot p \otimes e_2 \otimes b - a \otimes e_2 \otimes p \cdot b \\
&\qquad - a \cdot q \otimes e_1 \otimes b + a \otimes e_1 \otimes q \cdot b, \\
d_1(a \otimes e_1 \otimes b) &:= a \cdot p \otimes b - a \otimes p \cdot b, \\
d_1(a \otimes e_2 \otimes b) &:= a \cdot q \otimes b - a \otimes q \cdot b, \\
d_0(a \otimes b) &:= a \cdot b.
\end{aligned}
$$

- For $\hbar = 0$, this is the Koszul resolution of $\mathbb{C}[p, q]$.

The Hochschild cohomology results

Theorem

(P_k, d_k) is a projective A-bimodule resolution of A.
Let M be an A-bimodule. Then $\mathrm{HH}^n(A, M) = 0$ *for* $n \geq 3$ *and*

$$\mathrm{HH}^2(A, M) \cong M \;/\; [p, M] + [q, M].$$

In particular, $\mathrm{HH}^2(A, A) = 0$ *if* $\hbar \neq 0$.

Corollary

Any formal deformation quantisation of the Weyl algebra A with $\hbar \neq 0$ *is equivalent to the trivial one:* $m(a, b) = a \cdot b$ *for* $a, b \in A$.

Discontinuity of Hochschild cohomology for $\hbar \to 0$

- If $\hbar \neq 0$, then the Weyl algebra has the same Hochschild cohomology with coefficients A as $\mathbb{C}$:
  $\mathrm{HH}^0(A, A) = \mathbb{C} \cdot 1_A$ and $\mathrm{HH}^k(A, A) = 0$ for $k \geq 1$.
- If $\hbar = 0$, then the Hochschild cohomology of $A = \mathbb{C}[p, q]$ is much bigger:
  $\mathrm{HH}^k(A, A) = \{k\text{-vector fields on } \mathbb{R}^2\}$.

Part II: De Rham cohomology of smooth manifolds

- The final goal of this class is to define
  periodic cyclic cohomology,
  a cohomology theory for noncommutative algebras that generalises de Rham cohomology for smooth manifolds.
- Today we recall the definition of de Rham cohomology for smooth manifolds.

The first de Rham boundary map

Example (de Rham complex of $\mathbb{R}$)

$$\cdots \to 0 \to \mathrm{C}^\infty(\mathbb{R}) \to \mathrm{C}^\infty(\mathbb{R}) \to 0 \to \dots$$
$$f \mapsto f'.$$

The derivative map is surjective. Its kernel is $\mathbb{R} \cdot \{1\}$.
De Rham cohomology of $\mathbb{R}$: $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{R}) = 0$ and $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{R}) = \mathbb{R}$.

- Let M be any smooth manifold.
- The derivative of a smooth function $f\colon M \to \mathbb{R}$ at $x \in M$ is a linear map $\mathrm{T}_x M \to \mathrm{T}_{f(x)}\mathbb{R} \cong \mathbb{R}$.
- This defines a section of the cotangent bundle $\mathrm{T}^* M$.
- Let $\Omega^0(M) := \mathrm{C}^\infty(M)$ and let $\Omega^1(M)$ be the space of smooth sections of the cotangent bundle of M.
- $\mathrm{d}\colon \Omega^0(M) \to \Omega^1(M)$, $(\mathrm{d}f)(x) := D_x f$,
  is the first de Rham boundary map.

Physical interpretation

- Use a Riemannian metric to identify $\Omega^1(M)$ with the space of vector fields on M.
- Then d maps a smooth function V to its gradient $\mathrm{grad}(V)$.
- Physical interpretation: V is a potential function and $-\mathrm{grad}(V)$ is the force field defined by the potential.
- If $\mathrm{d}V = 0$, then V is locally constant (constant on each connected component of M). Thus the potential is unique up to adding a constant if M is connected.
- A force field is conservative if it admits a potential. This is equivalent to energy conservation.
- While every force field on $\mathbb{R}$ is conservative, this fails for all other smooth manifolds.

The second de Rham boundary

Proposition

A force field $\sum_{j=1}^n f_j \, \mathrm{d}x_j$ on $\mathbb{R}^n$ is conservative if and only if it satisfies the system of linear partial differential equations

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \qquad \forall 1 \leq i < j \leq n.$$

- Let $\Lambda^k(\mathrm{T}^* M)$ be the vector bundle on M defined at each point by taking the kth exterior power of $\Lambda^1(\mathrm{T}^* M)$.
- The second de Rham boundary on $\mathbb{R}^n$ is $\mathrm{d} \colon \Omega^1(\mathbb{R}^n) \to \Omega^2(\mathbb{R}^n)$, $\sum_{j=1}^n f_j \, \mathrm{d}x_j \mapsto \sum_{i,j=1}^n \frac{\partial f_j}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}x_j = \sum_{1 \leq i < j \leq n} \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right) \mathrm{d}x_i \wedge \mathrm{d}x_j$.

Proposition

The first de Rham cohomology of $\mathbb{R}^n$ vanishes for all $n \geq 1$.

The de Rham complex of a smooth manifold

- Let $\Omega^k(M)$ be the space of sections of $\Lambda^k(\mathrm{T}^*M)$ for $k \geq 0$.
- This vanishes for $k > \dim M$.
- Define the boundary map $\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)$ in local coordinates by

$$\mathrm{d}(f\,\mathrm{d}x_{i_1}\,\mathrm{d}x_{i_2}\ldots\mathrm{d}x_{i_k}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\,\mathrm{d}x_j\,\mathrm{d}x_{i_1}\,\mathrm{d}x_{i_2}\ldots\mathrm{d}x_{i_k}.$$

- Some work is needed to check that this is well defined and $\mathrm{d}^2 = 0$.

  The complex so defined is the de Rham complex.
- Its cohomology is called de Rham cohomology.

Properties of de Rham cohomology

Theorem

The de Rham cohomology and the singular cohomology of a smooth manifold with coefficients $\mathbb{R}$ are naturally isomorphic.

Corollary

De Rham cohomology is homotopy invariant:
Homotopic smooth maps $f, g\colon X \rightrightarrows Y$
induce the same map in de Rham cohomology.

Corollary (Poincaré Lemma)

$\mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}^n) = 0$ *for all* $k \geq 1$.

De Rham cohomology in dimension 3

- Let M be an oriented smooth manifold of dimension 3.
- $C^\infty(M) = \Omega^0(M)$.
- M has a volume form $\omega \in \Omega^3(M)$.
- $C^\infty(M) \cong \Omega^3(M)$, $f \mapsto f \cdot \omega$.
- $\Omega^2(M) \cong \Omega^1(M) \cong \mathfrak{X}(M)$, the space of smooth vector fields on M, by using a Riemannian metric on M and the resulting volume form.
- The de Rham complex becomes a cochain complex

$$0 \to C^\infty(M) \to \mathfrak{X}(M) \to \mathfrak{X}(M) \to C^\infty(M) \to 0.$$

- On $\mathbb{R}^3$:
  - $\mathrm{d}\colon \Omega^0(M) \to \Omega^1(M)$ is the gradient map.
  - $\mathrm{d}\colon \Omega^1(M) \to \Omega^2(M)$ is the rotation map.
  - $\mathrm{d}\colon \Omega^2(M) \to \Omega^3(M)$ is the divergence map.