Noncommutative Geometry IV: Differential Geometry

22. Noncommutative differential forms

R. Meyer

Mathematisches Institut
Universität Göttingen

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Noncommutative differential forms

- We are going to introduce the differential graded algebra $\Omega(A)$ of noncommutative differential forms over an algebra A.
- We describe each $\Omega^n(A)$ and $\Omega(A)$ by universal properties.
- We define the Hochschild boundary map $b\colon \Omega(A) \to \Omega(A)$.
- Hochschild homology is the homology of the chain complex $(\Omega(A), b)$.
- We explain how to compute Hochschild homology using projective bimodule resolutions.
- We carry this through for
  the algebra of polynomials and the Weyl algebra.

The bimodule of differential n-forms

Lemma

For $n \geq 1$, there is an A-bimodule $\Omega^n(A)$ such that for all A-bimodules M there is a natural bijection between $\mathrm{Hom}_{A,A}(\Omega^n(A), M)$ and normalised Hochschild n-cocycles $A^n \to M$.

Lemma

Let $\bar{A} := A/\mathbb{C} \cdot 1_A$. There are natural left A-module isomorphisms $\Omega^n(A) \cong A \otimes \bar{A}^{\otimes n}$ for all $n \in \mathbb{N}_{\geq 1}$, such that the right A-module structure on $\Omega^n(A)$ becomes

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \cdot b := a_0 \otimes a_1 \otimes \cdots \otimes (a_n \cdot b) \\ - a_0 \otimes a_1 \otimes \cdots \otimes (a_{n-1} \cdot a_n) \otimes b + a_0 \otimes a_1 \otimes \cdots \otimes (a_{n-2} \cdot a_{n-1}) \otimes a_n \otimes b \\ \mp \cdots + (-1)^n a_0 \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes b.$$

We rewrite $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ as $a_0 \, \mathrm{d}a_1 \ldots \mathrm{d}a_n$.

The universal Hochschild cocycles

Lemma

$$\Omega^n(A) := \Omega^1(A)^{\otimes_A n} := \underbrace{\Omega^1(A) \otimes_A \Omega^1(A) \otimes_A \cdots \otimes_A \Omega^1(A)}_{n \text{ factors}}$$

Definition

Elements of $\Omega^n(A)$ are called
noncommutative differential n-forms over A.

Proposition

The map $\mathrm{d}^{\cup n}\colon (a_1, \ldots, a_n) \mapsto \mathrm{d}a_1 \ldots \mathrm{d}a_n$ *is the*
universal normalised Hochschild n-cocycle:
it is a normalised Hochschild n-cocycle,
and any other normalised Hochschild n-cocycle
factors as $f \circ \mathrm{d}^{\cup n}$ *for a unique bimodule homomorphism* f.

Differential graded algebra structure

- A graded algebra is an algebra A together with a decomposition $A \cong \bigoplus_{n \in \mathbb{N}} A_n$ such that $A_n \cdot A_m \subseteq A_{n+m}$.
- A graded derivation on a graded algebra A is a linear map $d \colon A \to A$ with $d(A_n) \subseteq A_{n+1}$ that satisfies the graded Leibniz rule $d(a \cdot b) = d(a) \cdot b + (-1)^n a \cdot d(b)$ for $a \in A_n$, $b \in A_m$.
- A differential graded algebra is a graded algebra A with a graded derivation $d \colon A \to A$ that satisfies $d^2 = 0$.

The universal differential graded algebra over A

The bimodule structures on $\Omega^n(A)$ for $n \in \mathbb{N}$ extend to an associative multiplication

$$\Omega^n(A) \otimes \Omega^m(A) \to \Omega^{n+m}(A),$$
$$\omega \cdot a_0 \,\mathrm{d}a_1 \dots \mathrm{d}a_m := (\omega \cdot a_0)\,\mathrm{d}a_1 \dots \mathrm{d}a_m.$$

Proposition

The differential graded algebra $(\Omega(A), \mathrm{d})$ *is the universal differential graded algebra over* A*: any algebra homomorphism from* A *to the degree-0-part of a differential graded algebra* (B, D) *extends uniquely to a differential graded algebra homomorphism* $(\Omega(A), \mathrm{d}) \to (B, D)$*.*

Boundaries on differential forms

The differential d on $\Omega(A)$ is useless to get an analogue of de Rham cohomology:

Lemma

The cochain complex $(\Omega(A), \mathrm{d})$ *is homotopy equivalent to* $\mathbb{C}$ *in degree* 0.

Definition (Hochschild boundary)

$b \colon \Omega^{n+1}(A) \to \Omega^n(A)$,
$a_0\,\mathrm{d}a_1\,\mathrm{d}a_2 \dots \mathrm{d}a_{n+1} \mapsto (-1)^n[a_0\,\mathrm{d}a_1\,\mathrm{d}a_2 \dots \mathrm{d}a_n, a_{n+1}]$
$= a_0 a_1\,\mathrm{d}a_2 \dots \mathrm{d}a_{n+1} + \sum_{j=1}^{n} (-1)^j a_0\,\mathrm{d}a_1\,\mathrm{d}a_2 \dots \mathrm{d}(a_j a_{j+1}) \dots \mathrm{d}a_{n+1}$
$+ (-1)^{n+1} a_{n+1} a_0\,\mathrm{d}a_1 \dots \mathrm{d}a_n$.

Lemma

$b^2 = 0$.

Hochschild homology

Definition

The homology of the chain complex $(\Omega^n(A), b)$ is called the Hochschild homology $\mathrm{HH}_n(A)$ of the unital algebra A.

Definition

The commutator quotient $M/[A, M]$ of an A-bimodule M is the quotient of M by the linear span of $[a, m] := am - ma$ for all $a \in A$, $m \in M$.

Theorem

Let $P_\bullet \to A$ be a projective A-bimodule resolution of A. The Hochschild homology of A is naturally isomorphic to the homology of the chain complex of commutator quotients $P_\bullet \;/\; [P_\bullet, A]$.

Theorem

The Hochschild homology of $C^\infty M$ is defined by completing $\Omega^n(C^\infty M)$ suitably. Then $\mathrm{HH}_n(C^\infty M)$ is the space $\Omega^n(M)$ of differential n-forms on M.