Noncommutative Geometry IV: Differential Geometry

23. Towards periodic cyclic homology

R. Meyer

Mathematisches Institut
Universität Göttingen

Summer Term 2020

Towards periodic cyclic homology

- We are going to generalise de Rham cohomology to noncommutative algebras.
- We have seen that $\mathrm{HH}_k(A) = \mathrm{H}_k(\Omega^\bullet(A), b)$ generalises the space of de Rham k-forms $\Omega^k(M)$ on a smooth manifold M.
- We are going to define an operator $B\colon \Omega^k(A) \to \Omega^{k+1}(A)$.
- It is built out of d and b, and it anti-commutes with b.
- Then it induces maps $B_*\colon \mathrm{HH}_k(A) \to \mathrm{HH}_{k+1}(A)$ that generalise the de Rham boundary map.
- B_* is a coboundary map because $B^2 = 0$.
- The computations with b and B depend on the study of the operator $\kappa := 1 - bd - db$ and its minimal polynomial.
- We compute the cohomology of $(\mathrm{HH}_*(A), B_*)$ for polynomials in two variables and the Weyl algebra.

The Karoubi operator

- $b^2 = 0$ and $\mathrm{d}^2 = 0$, but b and d do not anticommute.
- $\kappa := 1 - [\mathrm{d}, b] = 1 - (\mathrm{d}b + b\mathrm{d})$ is called Karoubi operator.
- if $\omega \in \Omega^{n-1}(A)$, $x \in A$, then
  $$\begin{aligned}\kappa(\omega\,\mathrm{d}x) &= \omega\,\mathrm{d}x - (-1)^n\mathrm{d}([x,\omega]) - (-1)^{n+1}[x,\mathrm{d}\omega] \\ &= \omega\,\mathrm{d}x - (-1)^n[\mathrm{d}x,\omega] \\ &= (-1)^{n-1}\mathrm{d}x\cdot\omega\end{aligned}$$
- $B := \sum_{j=0}^{n} \kappa^j \circ \mathrm{d}$ on $\Omega^n(A)$
- $B(x_0\,\mathrm{d}x_1\ldots\mathrm{d}x_n) = \sum_{j=0}^{n} (-1)^{jn}\mathrm{d}x_j\ldots\mathrm{d}x_n\,\mathrm{d}x_0\ldots\mathrm{d}x_{j-1}$.

The (b, B)-bicomplex

Lemma
$b^2 = 0$, $B^2 = 0$, $[b, B] := bB + Bb = 0$.

- B maps the subspaces $\ker b$ and $\operatorname{im} b$ of $\Omega(A)$ into themselves.
- It induces a map $B_*\colon \mathrm{HH}_n(A) \to \mathrm{HH}_{n+1}(A)$ with $B_*^2 = 0$.

Question
Is the cohomology of $(\mathrm{HH}_\bullet(A), B_*)$ a good generalisation of de Rham cohomology?

Properties of κ

- κ is a chain map for b and d:
  $\kappa \mathrm{d} = \mathrm{d}\kappa$, $\kappa b = b\kappa$
- κ commutes with all operators built from b and d.
- $B^2 = 0$
- $\mathrm{d}\kappa^{n+1} = \kappa^{n+1}\mathrm{d} = \mathrm{d}$ on $\Omega^n(A)$.
- $\kappa B = B\kappa = B$.
- $1 + b\kappa^n \mathrm{d} = \kappa^n$ on $\Omega^n(A)$
- $\kappa^n b = b\kappa^n = b$ on $\Omega^n(A)$
- $\mathrm{d}b = 1 - \kappa^{n+1}$ and $b\mathrm{d} = \kappa^{n+1} - \kappa$ on $\Omega^n(A)$.
- $(\kappa^n - 1)(\kappa^{n+1} - 1) = 0$ on $\Omega^n(A)$.
- $Bb + bB = 1 - \kappa^n - \kappa^{n+1} + \kappa^{2n+1} = (1 - \kappa^n)(1 - \kappa^{n+1}) = 0$ on $\Omega^n(A)$.

Computation for polynomials

- $\mathrm{HH}_n(\mathrm{C}^\infty M) \cong \Omega^n(M)$ (if tensor products are completed)
- $$\mathrm{HH}_n(\mathbb{C}[x,y]) \cong \begin{cases} \mathbb{C}[x,y], & \text{if } n = 0, \\ \mathbb{C}[x,y]\,\mathrm{d}x \oplus \mathbb{C}[x,y]\,\mathrm{d}y, & \text{if } n = 1, \\ \mathbb{C}[x,y]\,\mathrm{d}x \wedge \mathrm{d}y, & \text{if } n = 2, \\ 0, & \text{if } n > 2. \end{cases}$$
- The action of B_* corresponds to the de Rham boundary map on differential forms.

Lemma

The cohomology of the cochain complex

$$0 \to \mathbb{C}[x,y] \xrightarrow{\mathrm{d}} \mathbb{C}[x,y]\,\mathrm{d}x \oplus \mathbb{C}[x,y]\,\mathrm{d}y \xrightarrow{\mathrm{d}} \mathbb{C}[x,y]\,\mathrm{d}x \wedge \mathrm{d}y \to 0$$

vanishes except in degree 0*, where it is* $\mathbb{C}$*.*

$$\mathrm{H}_n(\mathrm{HH}_*(\mathbb{C}[x,y]), B_*) \cong \begin{cases} \mathbb{C} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Computation for the Weyl algebra

- We have already described a small free bimodule resolution for the Weyl algebra $A := \mathbb{C}\langle p, q \mid [p, q] = \mathrm{i}\hbar\rangle$.
- We use it to compute $\mathrm{HH}_*(A)$:
  it is the homology of the chain complex
  $0 \to A \xrightarrow{(\mathrm{ad}_p, -\mathrm{ad}_q)} A \oplus A \xrightarrow{(\mathrm{ad}_q, \mathrm{ad}_p)} A \to 0.$
- $\mathrm{H}_n(\mathrm{HH}_*(A), B_*) \cong \mathrm{HH}_n(A) \cong \begin{cases} \mathbb{C} & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$
- Compare:
  $\mathrm{H}_n(\mathrm{HH}_*(\mathbb{C}[x, y]), B_*) \cong \begin{cases} \mathbb{C} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$