Noncommutative Geometry IV: Differential Geometry

24. Periodic cyclic homology

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Periodic cyclic homology

- We replace the cochain complex $(\mathrm{HH}_*(A), B_*)$ by a 2-periodic bicomplex with boundary map $b + B$.
- Its homology is called the periodic cyclic homology $\mathrm{HP}_*(A)$ of A.
- The cyclic homology groups $\mathrm{HC}_*(A)$ are non-periodic approximations to $\mathrm{HP}_*(A)$. They are related to Hochschild homology by a long exact sequence

$$\cdots \to \mathrm{HH}_n(A) \to \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \to \mathrm{HH}_{n-1}(A) \to \mathrm{HC}_{n-1}(A) \to \cdots .$$

- We compute the cyclic and periodic cyclic homology for the Weyl algebra and for algebras of smooth functions.
- We sketch the definition of a natural map from K-theory to periodic cyclic homology.
- We sketch how to define functionals $\mathrm{HP}_*(A) \to \mathbb{C}$ using closed graded traces on differential algebras over A.

The definition of periodic cyclic homology

$$\Omega(A) \quad \prod_{n=0}^{\infty} \Omega^n(A)$$
$$\Omega^{\text{even}}(A) \quad \prod_{n=0}^{\infty} \Omega^{2n}(A)$$
$$\Omega^{\text{odd}}(A) \quad \prod_{n=0}^{\infty} \Omega^{2n+1}(A)$$

Definition (periodic cyclic homology $\mathrm{HP}_*(A)$)

the homology of the 2-periodic chain complex

$$\cdots \to \Omega^{\text{even}}(A) \xrightarrow{B+b} \Omega^{\text{odd}}(A) \xrightarrow{B+b} \Omega^{\text{even}}(A) \xrightarrow{B+b} \Omega^{\text{odd}}(A) \to \cdots$$

A filtration on the periodic cyclic complex

- $\mathcal{F}_n := b(\Omega^n(A)) \times \prod_{k=n}^{\infty} \Omega^k(A)$ is a subcomplex of $(\Omega(A), b + B)$.
- These subcomplexes form a decreasing filtration with $\bigcap \mathcal{F}_n = 0$.

Theorem

Assume that $\mathrm{HH}_N(A) = 0$ *for all* $N \geq n$.
Then the chain complex $\mathcal{F}_n$ *is contractible.*
So $\mathrm{HP}_*(A)$ *is isomorphic to the homology of the truncated chain complex*

$$\Bigl(\prod_{k \in \mathbb{N}} \Omega^k(A) \,/\, \mathcal{F}_{n+1}, b + B\Bigr) \cong \Bigl(\prod_{k=0}^{n-1} \Omega^k(A) \times \frac{\Omega^n(A)}{b(\Omega^{n+1}(A))}, b + B\Bigr).$$

Cyclic homology $\mathrm{HC}_*(A)$

The following diagram anti-commutes:

$$\begin{array}{ccccccc}
\dots & \longleftarrow & \dots & \longleftarrow & \dots & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^3(A) & \xleftarrow{B} & \Omega^2(A) & \xleftarrow{B} & \Omega^1(A) & \xleftarrow{B} & \Omega^0(A) \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
\Omega^2(A) & \xleftarrow{B} & \Omega^1(A) & \xleftarrow{B} & \Omega^0(A) & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & & & \\
\Omega^1(A) & \xleftarrow{B} & \Omega^0(A) & & & & \\
\downarrow{\scriptstyle b} & & & & & & \\
\Omega^0(A) & & & & & &
\end{array}$$

The homology of the resulting total complex is $\mathrm{HC}_*(A)$.

The degree n space in this total complex is

$$\Omega^n A \times \Omega^{n-2} A \times \Omega^{n-4} \times \cdots ,$$

the boundary is $b + B$ on most summands, and just b on the first.

From Hochschild to cyclic homology

Theorem

There is a long exact sequence of homology groups

$$\cdots \to \mathrm{HH}_n(A) \to \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \\ \to \mathrm{HH}_{n-1}(A) \to \mathrm{HC}_{n-1}(A) \to \cdots . \quad (1)$$

Theorem

If $\mathrm{HH}_n(A) = 0$ *for all* $n > N$,
then $\mathrm{HP}_N(A) = \mathrm{HC}_N(A)$ *and* $\mathrm{HP}_{N+1}(A) = \mathrm{HC}_{N+1}(A)$.

Sample computations

Example (Weyl algebra A)

$\mathrm{HC}_n(A) = 0$ for $n \leq 1$ and $\mathrm{HC}_n(A) \cong \mathbb{C}$ for $n \geq 2$,
with an invertible map $S\colon \mathrm{HC}_{n+2}(A) \to \mathrm{HC}_n(A)$ for $n \geq 2$.
$\mathrm{HP}_0(A) = \mathbb{C}$, $\mathrm{HP}_1(A) = 0$.

Theorem

Let M be a smooth manifold. Let $n \in \mathbb{N}$. Then

$$\mathrm{HC}_n(\mathrm{C}^\infty M) \cong \Omega^n M/\mathrm{d}(\Omega^{n-1} M) \oplus \mathrm{H}_{\mathrm{dR}}^{n-2}(M) \oplus \mathrm{H}_{\mathrm{dR}}^{n-4}(M) \oplus \cdots,$$

$$\mathrm{HP}_n(\mathrm{C}^\infty M) \cong \bigoplus_{k \in \mathbb{Z}} \mathrm{H}_{\mathrm{dR}}^{n-2k}(M).$$

Link to K-theory

Definition

Let A be a unital Banach algebra.
The group $\mathrm{K}_0(A)$ consists of homotopy classes of idempotent elements in $\mathbb{M}_n A$ for $n \in \mathbb{N}$.
The group $\mathrm{K}_1(A)$ consists of homotopy classes of invertible elements in $\mathbb{M}_n A$ for all $n \in \mathbb{N}$.

Theorem

There is a natural group homomorphism $\mathrm{K}_*(A) \to \mathsf{HP}_*(A)$, called the Chern–Connes character.

Construction of the Chern–Connes character

- An idempotent element in $\mathbb{M}_n A$ is a non-unital homomorphism $\mathbb{C} \to \mathbb{M}_n A$.
- An invertible element in $\mathbb{M}_n A$ is a unital homomorphism $\mathbb{C}[t, t^{-1}] \to \mathbb{M}_n A$.
- $\mathrm{HP}_0(\mathbb{C}) = \mathbb{C}$, $\mathrm{HP}_1(\mathbb{C}[t, t^{-1}]) \cong \mathbb{C}$
- $\mathrm{HP}_*(\mathbb{M}_n A) \cong \mathrm{HP}_*(A)$
- HP_* is functorial for non-unital homomorphisms.

Periodic cyclic cocycles from closed graded traces

- Let (C, ∂) be a differential graded unital algebra and let $\varphi\colon A \to C_0$ be a unital algebra homomorphism from A to the degree-zero part of C.
  Let $\tau\colon C_n \to \mathbb{C}$ be a linear map that is a closed graded trace.
- $\varphi_*\colon \Omega(A) \to C$ is a differential graded algebra homomorphism.
- $\tau' := \tau \circ \varphi_*\colon \Omega^n(A) \to \mathbb{C}$ is a closed graded trace.
- $\tau' \circ \mathrm{d} = 0$, $\tau' \circ b = 0$, $\tau' \circ \kappa = \tau'$, $\tau' \circ B = 0$.
- Thus τ' induces a map $\mathrm{HP}_n(A) \to \mathbb{C}$.