Leavitt path algebras as Cohn localisations and their Hochschild homology

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1 Introduction

In mathematics, promising generalisations often evolve from tractable counterexamples for well-studied structural properties. It motivates to include them in the existent theory beyond the scope of this violated convenient property with as much consistency as possible. The generalisations from smooth functions to Borel measurable functions, from commutative algebras to noncommutative algebras, or from unital rings to a similarly well-behaved familiy of possibly nonunital rings are examples of this theme. Likewise, also the historical route towards Leavitt path algebras in their current form originates at counterexamples. Here, it concerns the property of an invariant basis number that is present in classical linear algebra in terms of an implication like

$$m, n \in \mathbb{N} \colon \mathbb{R}^m \cong \mathbb{R}^n \implies m = n.$$

In the 1950s, Leavitt introduced a universal algebra $L(1, n)$ that is isomorphic to its own n-th power in the sense that there is an $L(1, n)$-bimodule isomorphism $L(1, n) \cong L(1, n)^n$. Starting from a free associative k-algebra in n generators one arrives there by postulating that the $(n \times 1)$-matrix

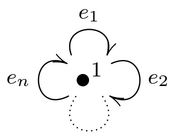
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot () \colon k[x_1, \ldots, x_n]^+ \to (k[x_1, \ldots, x_n]^+)^n$$

that multiplies jointly with all generators becomes invertible. For this, the entries $(x_j^*)_{j \leq n}$ that make up the inverse $(1 \times n)$-matrix have to be adjoined appropriately. This procedure reminds us of a classical localisation in commutative ring theory, but on the level of matrix equations. In fact, it is an instance of the so called Cohn localisation at a matrix multiplication map treated in Section 2.4.

However, so far the word “path” is still missing. Apart from the free algebra over a set with n elements and no predefined structure, there are analogous relatively free constructions for mathematical objects that already admit some internal structure that has to be reflected. They are discussed in Section 2.2 and also apply to countable directed graphs

$$E = (E^0, E^1, s \colon E^1 \to E^0, r \colon E^1 \to E^0)$$

or quivers, for short. This leads to the notion of a path algebra kE that is designed to model vertices as orthogonal idempotents and edges as almost free generators. Their only relation is to be invariant under multiplication by their respective source or range vertex. The original free algebra with n generators is included in this picture as the path algebra of the quiver

$R_n =$ 

with one vertex and n loops attached to it.
Actually, path algebras prove to belong to the quite restrictive class of quasi-free algebras treated in Section 2.3. Similarly to the wider notion of projective modules compared to free ones, quasi-freeness provides a suitable generalisation of free algebras as well. In fact, the approach above of taking the Cohn localisation at a joint multiplication map also passes on to path algebras. If there are finitely many edges that start from a given vertex v, it corresponds to the localisation at a joint prolongation map of paths that end in v:

$$\varphi^v = (\cdot e)_{e \in s^{-1}(v)} \colon kEv \to \bigoplus_{e \in s^{-1}(v)} kEr(e).$$

Again, this leads to additional generators $(e^*)_{e \in s^{-1}(v)}$ that are designed to form the inverse multiplication map. It suggests to import "adjoints" not only for those edges starting at v, but for all of them by localising at suitable multiplication maps.
Historically, however, with the rise of noncommutative geometry in the 1980s, a more analytic approach takes over at this point and the concept of a graph C*-algebra is developed first. A graph C*-algebra is motivated by operator algebraic means and by its strength to reflect aspects of theoretical physics. Instead of localisation, the idea for its construction is originally based on the theory of projections and $\{0, 1\}$-valued matrices. Vertices are modelled as orthogonal projections and edges as associated partial isometries with well-behaved range and source projections. If one drops the analytic aspect of the completion in a C*-norm to restrict attention to the involved *-algebraic Cuntz-Krieger relations, this modelling provides Leavitt path algebras instead.
In fact, Leavitt path algebras are known under this name since 2005 and gained attention by the research groups around both Abrams and Ara. They have been studied intensely and much of their algebraic structure has been worked out since then.
However, this historic C*-algebraic route neglects the localisation idea that has been the initial motivation for Leavitt algebras after all. This thesis therefore aims to use Cohn

localisations as an alternative and natural way to obtain Leavitt path algebras.
For this, it starts off with non-unital module theory in Section 2.1 that paves the way to treat countable quiver inputs while the rest of the second chapter provides a detailed approach towards the previously mentioned algebra constructions.
It continues with the localisation idea that is pointed out above and treated in more detail in Section 3.1. Ultimately, the goal is to pose the Cuntz-Krieger relations as invertibility postulations for suitable path prolongation maps. This is done in Section 3.2 and justifies to treat any relative Leavitt path algebra L as a Cohn localisation of a somewhat generalised path algebra B that is quasi-free:

$$L = \mathrm{Cohn}(B).$$

One of the key advantages of this interpretation is that quasi-freeness is conserved under Cohn localisations. This allows to effectively reduce the computational effort for the bimodule of noncommutative forms that is introduced in Section 4.1 to the conceptually easier path algebra B. These computations are carried out in Section 4.3.
Therefore, the Cohn localisation picture turns out to be useful both for structural and for computational purposes. It leads to a projective bimodule resolution of length one for L without much theory in Section 4.2. This result can in turn be used as a tool for further homological computations such as for Hochschild homology in Section 5.1 or also for periodic cyclic homology in Section 5.2.
In fact, for row-finite quivers, these results have already been established by different methods. On the one hand, Chen, Xi and Wang worked out an explicit projective bimodule resolution in [9], which relies on relative bimodules of noncommutative forms. On the other hand, Ara and Cortiñas also computed the Hochschild homology in [5] as a side-product of their K-theoretical studies in [6], which both rely on the theory of crossed products as well.
Nevertheless, this thesis provides an alternative route towards these results and even extends them to the nonunital framework of all countable quivers. It therefore indicates the strength of the Cohn localisation picture in this context and advertises to treat relative Leavitt path algebras as Cohn localisations more often.

2 Theoretic foundation

First, let us introduce the algebraic objects that we plan to use throughout this thesis.
A ring R is in general not required to be unital and a module M over it is given by an additively written abelian group with a compatible multiplicatively written R-module structure. A module over a unital ring is called unital, if the unit acts as the identity. If necessary, left, right or bimodules are highlighted by a corresponding index notation ${}_RM$, M_R or ${}_RM_R$, respectively. By passing to the opposite ring, left and right modules can be treated along the same lines. Thus, we keep speaking of one-sided modules and for the sake of simplicity definitions often only refer to one of both possibilities.
The same holds for an algebra A, which is defined with respect to a given field k in the sense that it is a k-vector spaces with an additional k-bilinear and associative multiplication. An algebra also does not necessarily have to have a unit.
In order to benefit from many advantages of unital algebras and unital modules, though, we often work with local units or at least with self-induced algebras and smooth modules. These concepts generalise the unital framework to some extent. See also [21].

2.1 Idempotents and nonunital rings

Definition 2.1 ([19, pp. 138-139])**.** A ring R is called *self-induced* if the canonical multiplication map induces an isomorphism between the one-sided balanced tensor product

$$R \otimes_R R := \frac{R \otimes R}{\operatorname{span}\{xr \otimes y - x \otimes ry \mid x, y, r \in R\}}$$

and R itself via:

$$\begin{aligned} \operatorname{mult}\colon R \otimes_R R &\to R, \\ r \otimes_R s &\mapsto rs. \end{aligned}$$

A module ${}_RM$ over a self-induced ring R is called *smooth* if the R-multiplication induces an isomorphism $R \otimes_R M \cong M$. Smoothness of right or bimodules is defined analogously. Note that a ring R is self-induced if and only if it is smooth as a bimodule over itself. See also [19] for the construction of balanced tensor products in general.

Remark 2.2 ([23, Def. 3.2]). Self-induced rings and smooth modules over them prove to be quite well-behaved in the sense that most arguments from the tensor calculus for unital

rings and modules carry over to them.
In bicategorical language, they generalise the bicategory of unital rings R. More concretely, the bicategory in question has unital rings as objects, unital bimodules ${}_R M_S$ as arrows $S \to R$ and bimodule homomorphisms as 2-arrows. The "composition" of arrows is given by taking balanced tensor products and is well-defined up to the canonical isomorphisms of tensor calculus. Now, the concept of self-induced rings and smooth modules extends this framework in such a way that the bimodule ${}_R R_R$ still serves as a unit arrow at R. The demanded bimodule isomorphisms showing up in Definition 2.1 are precisely the natural uniter 2-arrows that justify the interpretation of ${}_R R_R$ as a unit arrow in this framework. The concept of the adjoint pair of restriction and extension of scalars still works out, too. See also [23, Def. 3.2.3] and [18, pp. 144 f.].

Smooth modules also effectively generalise the postulate that a unit element has to act trivially as long as such an element exists in the underlying ring. This can be made precise with the concept of non-degeneracy.

Definition 2.3. A module ${}_R M$ over a ring R is called *non-degenerate* if it is spanned by the R-action. With the shorthand span notation this reads as

$$RM := \operatorname{span}\{rm \mid r \in R,\, m \in M\} = M.$$

Lemma 2.4 ([24, Lemma 5.6])**.** *A module ${}_R M$ over a unital ring R is non-degenerate if and only if $1_R \in R$ acts as the identity on M.*

Proof. $1_R \cdot m = m$ for all $m \in M$ directly implies $RM = M$. Conversely, $1_R \cdot (rm) = (1_R \cdot r)m = rm$ for $r \in R$, $m \in M$ shows that 1_R automatically acts as the identity on RM. □

Remark 2.5. Note that a smooth R-module M is automatically non-degenerate since its corresponding multiplication map is surjective. So, in some sense, we can say that a smooth module both admits a global unitality condition inspired by Lemma 2.4 and a local invariance condition coming from injectivity of the multiplication map.

Lemma 2.6. *Let M be a smooth bimodule over a self-induced ring R. Then its commutator quotient*

$$M_\# = {}^{M}\!\big/\!_{[R, M]} := \frac{M}{\operatorname{span}\{[r, m] = rm - mr \mid r \in R,\, m \in M\}}$$

is naturally isomorphic to the two-sided balanced tensor product

$$M \otimes_{RR} R := \frac{M \otimes R}{\operatorname{span}\{rms \otimes t - m \otimes str \mid r, s, t \in R,\, m \in M\}}.$$

Proof. The balanced tensor product already allows to shift ring elements next to $\otimes_R$. We therefore deal with a quotient module of $M \otimes_R R \cong M$. Since the module is assumed to be smooth with $M = MR$, this quotient module is in fact the commutator quotient:

$$\begin{aligned} M \otimes_{RR} R &= \frac{M \otimes_R R}{\operatorname{span}\{rm \otimes_R c - m \otimes_R cr \mid r, c \in R,\, m \in M\}} \\ &\cong \frac{M}{\operatorname{span}\{rmc - mcr = [r, mc] \mid r, c \in R,\, m \in M\}} \\ &= {}^{M}\!\big/_{[R, M]}. \end{aligned}$$

□

In fact, the existence of local units already allows to use the isomorphisms mentioned above. Roughly speaking, the fact that we are dealing with finitely many elements at the same time enables us to work in a unital subring depending on them. For a concrete shape of these unital subrings we introduce idempotent elements.

Definition 2.7 ([7, Def. 4.1.1])**.** An element of a ring $e \in R$ is called an *idempotent* if $e^2 = e$. We also refer to the set of idempotents as $\operatorname{Idem}(R)$. One-sided multiplication by e on R yields the so called *corner* subrings eR and Re. By design, $e = e^2 \in eR \cap Re$ acts trivially on them from the respective side. This invariance condition serves as the defining property for the corner because we have $r = er' \in eR$ if and only if $er = e^2r' = er' = r$. The same construction with another $f \in \operatorname{Idem}(R)$ also allows for two-sided corners eRf. Two idempotents e and f are called *orthogonal* ($e \perp f$) if $ef = fe = 0$. The sum of orthogonal idempotents $e \perp f$ is also an idempotent:

$$(e + f)^2 = e^2 + ef + fe + f^2 = e^2 + f^2 = e + f.$$

Furthermore, we introduce a partial order on the set of idempotents by saying that $e \leq f$ if $ef = fe = e$. In this case, $f - e$ is again an idempotent:

$$(f - e)^2 = f^2 - fe - ef + e^2 = f^2 - 2e + e^2 = f - e.$$

In the unital case, all idempotents e satisfy $(1 - e) \perp e$ and $0 \leq e \leq 1$.

Definition 2.8 ([2, Def. 1.2.10])**.** A ring R is said to have a set of *local units* $U \subseteq \mathrm{Idem}(R)$ if every finite subset is contained in a unital corner uRu for some $u \in U$. If U is spanned by a family of pairwise orthogonal idempotents, R is said to have *enough idempotents.*

Lemma 2.9. *Let A be an algebra that admits a set of local units. Then A is self-induced and an A-module M_A is smooth if and only if every $m \in M$ is invariant under a local unit in A.*

Proof. Let $a \in A$. Note that tensoring by any local unit u_a provides a lift of $a \in A$ under mult. Indeed, $\mathrm{mult}(a \otimes_A u_a) = au_a = a$. This tensoring is even independent of our choice of the local unit since for any two local units u, u' for a we observe that

$$a \otimes_A u - a \otimes_A u' = a \otimes_A (u - u') = a(u - u') \otimes_A v = 0 \otimes_A v = 0$$

if v in turn is a local unit for $u - u'$. Therefore, we obtain a well-defined splitting map $s\colon A \to A \otimes_A A$. Since the local unit can always be chosen jointly for all involved elements such as $a, b, x, y \in A$, this map is both k-linear for $\lambda \in k$ and an A-bimodule map:

$$\begin{aligned} s(\lambda a + b) &= (\lambda a + b) \otimes_A u_{a,b} = \lambda(a \otimes_A u_{a,b}) + b \otimes_A u_{a,b} = \lambda s(a) + s(b), \\ s(xay) &= xay \otimes_A u_{x,a,y} = xa \otimes_A u_{x,a,y} y u_{x,a,y} = xa \otimes_A u_{x,a,y} y = xs(a)y. \end{aligned}$$

Finally, it is also right inverse to mult and the claimed isomorphism holds:

$$s(\mathrm{mult}(a \otimes_A b)) = \mathrm{mult}(a \otimes_A b) \otimes_A u_{ab} = u_a ab \otimes_A u_{ab} = u_a \otimes_A ab = a \otimes_A b.$$

If every element in an A-module $m \in M_A$ is invariant under a local unit $u(m) \in U$, we prove along the same lines with m in place of a that tensoring with this idempotent establishes the smoothness isomorphism $M \otimes_A A \cong M$. Conversely, a smooth module is non-degenerate, which allows to write any element $m \in M = MA$ as

$$m = \sum_{j \leq n} m_j a_j, \quad a_j \in A,\ m_j \in M.$$

If u is a local unit for these a_j, then m is also invariant by design, that is, $mu = m$. □

Example 2.10 ([20, p. 32]). For $n \in \mathbb{N}$, let $M_n(R)$ denote the ring of $(n \times n)$-matrices over a ring R with entrywise addition and matrix multiplication. If $R = A$ is a k-algebra, then so is $M_n(A)$ with entrywise scalar multiplication.

Finite matrix rings embed into the ring of finitely supported matrices with index set $\mathbb{N} \times \mathbb{N}$, which can be written as a union

$$M_\infty(R) := \bigcup_{n \in \mathbb{N}} M_n(R)$$

by interpreting $M_n(R)$ as the upper-left $(n \times n)$-corner with zeros elsewhere. If R is unital, then also $M_n(R)$ is unital with the unit matrix $1_n = \operatorname{diag}(1, \dots, 1)$. In this case, also $M_\infty(R)$ has enough idempotents given by the family of pairwise orthogonal diagonal elementary matrices $\{E_{jj} = (\delta_{(k,l),(j,j)})_{(k,l) \in \mathbb{N}^2} \mid j \in \mathbb{N}\}$, but it is nevertheless nonunital due to the fact that the formal $\mathbb{N} \times \mathbb{N}$ unit matrix

$$\operatorname{diag}(1, 1, \dots) = \sum_{j \in \mathbb{N}} E_{jj}$$

is no longer finitely supported.
The matrix ring $M_\infty(R)$ still acts by matrix multiplication on the ring of R-valued sequences $R^\mathbb{N}$. For $r = (r_n)_{n \in \mathbb{N}} \in R^\mathbb{N}$ and $A = (a_{ij})_{i,j \in \mathbb{N}} \in M_\infty(R)$, the image sequence

$$(Ar)_i := \sum_{j \in \mathbb{N}} a_{ij} r_j$$

is well-defined since all involved sums are finitely supported by design. If $n \in \mathbb{N}$ is chosen minimal such that $A \in M_n(R)$, then the tail of the sequence $(r_j)_{j>n}$ has no impact on Ar at all, which itself vanishes for entries $i > n$. Hence, matrix multiplication maps into the ring of finitely supported R-valued sequences $R^\infty := \bigcup_{n \in \mathbb{N}} R^n$. In this context, we interpret n-tuples in R^n as sequences that become zero for $m > n$. To sum up, $M_\infty(R)$ and R^∞ allow to combine matrix theory for n-tuples of finite length. Linear algebra over them always reduces to the finite-dimensional case for a large enough n.

Lemma 2.11. *Let R be a ring and let $p, q \in \operatorname{Idem}(M_\infty(R))$ be idempotent matrices over R. Then $\varphi\colon pR^\infty \to qR^\infty$ is an R-module homomorphism if and only if $\varphi = X \cdot ()$ acts by left matrix multiplication for some $X \in qM_\infty(R)p$. By passing to the opposite ring, an analogous statement holds for left corner modules and matrix multiplication from the right.*

Proof. Choose $n \in \mathbb{N}$ large enough such that $p, q \in M_n(R)$. As Example 2.10 shows, this restricts attention to maps $pR^\infty = pR^n \to qR^\infty = qR^n$ and the matrix corner $qM_\infty(R)p = qM_n(R)p$. Now, matrix multiplication by $X \in qM_n(R)p$ yields a well-defined map $pR^n \to$

qR^n since $r \in pR^n$ implies $Xr = qXr \in qR^n$. Entrywise additivity and distributivity in R ensure that it is even an R-module homomorphism.
Conversely, any R-module homomorphism $\varphi\colon pR^n \to qR^n$ is pinned down by its values on the columns of p, called $p^{(j)} \in pR^n$ for $j \le n$. To see this, first note that $p^2 = p$ ensures that the columns actually are in pR^n because

$$pp^{(j)} = \left(\sum_{k\le n} p_{ik}p_k^{(j)}\right)_{i\le n} = \left(\sum_{k\le n} p_{ik}p_{kj} = p_{ij}\right)_{i\le n} = p^{(j)}.$$

So φ can be applied and allows to define a matrix X columnwise by

$$X^{(j)} := \varphi(p^{(j)}).$$

The claim is that φ acts by matrix multiplication with X. Indeed, since φ is a homomorphism, we observe for any $r = (r_j)_{j\le n} \in pR^n$ that

$$\varphi(r) = \varphi(pr) = \varphi\left(\sum_{j\le n} p^{(j)}r_j\right) = \sum_{j\le n}\varphi(p^{(j)})r_j = \sum_{j\le n} X^{(j)}r_j = Xr.$$

Furthermore, a columnwise computation for $k \le n$ shows the desired invariance properties:

$$\begin{aligned}(Xp)^{(k)} &= \sum_{j\le n}\varphi(p^{(j)})p_{jk} = \varphi\left(\sum_{j\le n} p^{(j)}p_{jk}\right) = \varphi(p^{(k)}) = X^{(k)},\\ (qX)^{(k)} &= \sum_{j\le n} q^{(j)}\varphi(p^{(k)})_j = \varphi(p^{(k)}) = X^{(k)}.\end{aligned}$$ □

In contrast to self-inducedness or even the existence of local units, which enable to transfer properties of a unital ring $(R, 1_R)$ to some extent, another tool to deal with nonunital rings, or more often nonunital algebras, is to adjoin a unit element.

Definition 2.12 ([13, Def. 1.35])**.** Let R be a ring. The abelian group $R^+ := R \oplus \mathbb{Z}$ equipped with the ring structure $(r, n)(s, m) := (rs + ns + rm, nm)$ for $r, s \in R$ and $n, m \in \mathbb{Z}$ is called the *unitalisation* of R. The element $1 := (0, 1)$ serves as a unit element and the second coordinate embedding $\mathbb{Z} \to R^+$ provides the unique unital homomorphism that splits the ring extension

$$R \rightarrowtail R^+ \twoheadrightarrow \mathbb{Z}\,.$$

Every R-module becomes a unital R^+-module in a unique way by declaring that 1 acts as

the identity.
If $R = A$ is even a k-algebra, we take the k-vector space $A^+ := A \oplus k$ equipped with the algebra structure $(a, \lambda)(b, \mu) := (ab + \lambda b + a\mu, \lambda\mu)$ for $a, b \in A$ and $\lambda, \mu \in k$ as its unitalisation instead.

Remark 2.13. In principle, also a unital ring R or algebra can be unitalised. However, the image $1_R = (1_R, 0)$ of the unit element under the canonical inclusion of R no longer acts as a unit element on R^+ like $1 = (0, 1)$ does. For $(r, \lambda) \in R^+$, we rather have that

$$1_R \cdot (r, \lambda) = (r + \lambda 1_R, 0) = (r, \lambda) \cdot 1_R.$$

So it merely becomes an idempotent $1_R \leq 1 = (0, 1)$ and the unitalisation can be seen as $R^+ \cong R \oplus \mathbb{Z}$.

Throughout, exact sequences play an important role for the analysis of algebraic objects. Hence, it is also of interest whether, or to what extent, a functor preserves exact sequences.

Proposition 2.14 ([18, p. 23])**.** *Let M be a bimodule over a ring R. Then the representable functor*

$$\begin{aligned} \operatorname{Hom}_{R,R}(M, -)\colon \operatorname{Mod}_{R,R} &\to \mathrm{Ab} \\ (f\colon S \to T) &\mapsto \left(f_*\colon \operatorname{Hom}_{R,R}(M, S) \to \operatorname{Hom}_{R,R}(M, T)\right), \end{aligned}$$

which acts by postcomposition on arrows, is left-exact.

Proof. Consider an exact sequence of R-bimodules:

$$0 \longrightarrow S \xrightarrow{\sigma} T \xrightarrow{\tau} U \longrightarrow 0\,.$$

Then the sequence

$$0 \longrightarrow \operatorname{Hom}_{R,R}(M, S) \xrightarrow{\sigma_*} \operatorname{Hom}_{R,R}(M, T) \xrightarrow{\tau_*} \operatorname{Hom}_{R,R}(M, U)$$

is exact at the first spot if and only if σ_* is injective. This follows immediately from the observation that any $f \in \operatorname{Hom}_{R,R}(M, S)$ with $\sigma_*(f) = 0$ satisfies $\sigma(f(m)) = 0$ and therefore $f(m) = 0$ for all $m \in M$. At the middle spot, $\tau_*\sigma_* = (\tau\sigma)_* = 0_*$ ensures the inclusion $\operatorname{Im}(\sigma_*) \subseteq \ker(\tau_*)$. The converse inclusion is checked by a quick diagram chase for $g \in \ker(\tau_*)$. By assumption, for any $m \in M$ we have that $g(m) \in \ker(\tau) = \operatorname{Im}(\sigma)$.

Hence, injectivity of σ yields a unique preimage $s \in S$ and $f(m) := s$ defines a lifting homomorphism with $\sigma_*(f) = g$. $\square$

Remark 2.15 ([12, pp. 156-162]). The previous proof also works analogously for just one-sided modules and includes the treatment of $R = A$ being a k-algebra. In any of these setups, however, we cannot expect the Hom-functor to be right exact and therefore exact in general. In the notation of Proposition 2.14, exactness of $\mathrm{Hom}_{R,R}(M, -)$ at the third spot corresponds to the property that τ^* is surjective. Spelled out, this means that for any surjective $\tau \in \mathrm{Hom}_{R,R}(T, U)$ we can write $h \in \mathrm{Hom}_{R,R}(M, U)$ as $h = \tau g$ for some $g \in \mathrm{Hom}_{R,R}(M, T)$. Phrased diagrammatically:

$$\begin{tikzcd} & M \arrow[dl, dotted, "\exists g"'] \arrow[d, "h"] \\ T \arrow[r, two heads, "\tau"] & U. \end{tikzcd}$$

A standard category theoretical argument that involves a pullback construction shows that it already suffices to check this property for $U = M$ and $h = \mathrm{Id}_M$. The corresponding proof idea is the same as the one we will meet later in Theorem 2.36.

This motivates the following definition.

Definition 2.16 ([18], [12, pp. 156-162]). Let R be a ring. An R-module P is called *projective* if its associated Hom-functor $\mathrm{Hom}(P, -)$ is exact. Equivalently, this happens if and only if any surjective module homomorphism $\tau \colon T \to P$ splits by a module homomorphism. That is, there is a module homomorphism $s \colon P \to T$ with $\tau \circ s = \mathrm{Id}_P$:

$$\begin{tikzcd} T \arrow[r, two heads, "\tau"] & P. \arrow[l, dotted, bend left, "\exists\, \mathrm{hom}\, s"] \end{tikzcd}$$

Remark 2.17 ([18, p. 24]). Intuitively, a projective module allows to lift any outgoing homomorphism into a quotient module along the quotient map. In fact, it turns out to be a restrictive property. Even at the level of abelian groups, that is, for $R = \mathbb{Z}$, the cyclic groups $M = \mathbb{Z}/m\mathbb{Z}$ for $m \geq 2$ provide a family of counterexamples. Every homomorphism $\varphi \colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}$ satisfies $m\varphi([1]) = \varphi([m]) = 0$ and thus has to be the zero homomorphism, which makes it impossible to lift the identity on $\mathbb{Z}/m\mathbb{Z}$ along the natural quotient map.

Projective modules form an important class of algebraic objects. Conveniently, we have that direct sums of projective modules P_i are again projective since the Hom-functor

$\mathrm{Hom}(\oplus_i P_i, -) = \prod_i \mathrm{Hom}(P_i, -)$ factors into the component Hom-functors that are known to be exact. Over rings with local units, the finitely generated projective modules can even be concretely classified up to isomorphism by using idempotents of matrix rings. They turn out to be the corners in R^∞ that we have studied above in Example 2.10 and Lemma 2.11.

Definition 2.18 ([13, Def. 1.2])**.** Let R be a ring and let M_R be a non-degenerate module over it. It is called *finitely generated* if there are finitely many $x_1, \ldots, x_n \in M$ such that

$$\begin{aligned} R^n &\to M \\ (a_1, \ldots, a_n) &\mapsto \sum_{j \leq n} x_j a_j \end{aligned}$$

is a surjective module homomorphism. In the span notation, this reads as

$$\{x_i \mid i \leq n\}R = \mathrm{span}\{x_i r \mid i \leq n, r \in R\} = M.$$

If R has local units, then M is smooth by Lemma 2.9.

Lemma 2.19 ([13, Lemma 1.8])**.** *Let R be a ring with local units. Then any finitely generated projective R-module M_R is isomorphic to some eR^∞ for an idempotent $e \in M_\infty(R)$. Conversely, all such modules are finitely generated and projective.*

Proof. Since M_R is finitely generated, there is a surjective module homomorphism $\pi\colon R^n \twoheadrightarrow M$. By projectivity this map splits by a homomorphism $\iota\colon M \to R^n$ and provides a split extension $\ker(\pi) \rightarrowtail R^n \twoheadrightarrow M$. This argument shows that finitely generated projective modules are direct summands of R^∞. More concretely, M is isomorphic to the range of $\iota \circ \pi\colon R^n \to R^n$. Since we know from Lemma 2.11 that one-sided module endomorphisms $R^n \to R^n$ are given by matrix multiplication maps with some $n \times n$ matrix on the opposite side, $\iota \circ \pi$ can be described by matrix multiplication with an idempotent $e \in M_n(R)$ because $(\iota \circ \pi) \circ (\iota \circ \pi) = \iota \circ \mathrm{Id}_M \circ \pi$. Thus, we obtain an isomorphism $M \cong eR^n = eR^\infty$, as claimed. Conversely, for any idempotent $e \in M_\infty(R)$ there is an $n \in \mathbb{N}$ such that we can interpret it as an idempotent $e \in M_n(R)$. This gives that $eR^\infty = eR^n$ is finitely generated by the columns $(e^{(j)})_{j \leq n}$. So it just remains to show projectivity. At this point, we can pass to the unitalisation since $e \cdot (\delta_{ij})_{i \leq n} = e^{(j)} \in eR^n$ implies $e(R^+)^n = eR^n$ and make use of the direct sum decomposition $(R^+)^n = (1_n - e)(R^+)^n \oplus e(R^+)^n$. Given a surjective homomorphism $p\colon N \twoheadrightarrow eR^n$ we can choose a preimage $n^{(j)}$ for the columns $e^{(j)}$. If we now use freeness of

$(R^+)^n$ to extend the map $(\delta_{i,j})_{i\leq n} \mapsto n_j$ to a homomorphism $g'\colon (R^+)^n \to N$, we observe that $p \circ g' = e \cdot ()\colon (R^+)^n \to eR^n$ acts trivially on $e(R^+)^n$. As a result, if we restrict g' to eR^n by precomoposing with the inclusion map $\iota\colon e(R^+)^n \to (R^+)^n$, then $g = g' \circ \iota$ provides the desired section for p. □

Corollary 2.20. *Let R be a ring with local units. Then both R_R and ${}_RR$ are direct limits of finitely generated projective modules.*

Proof. It suffices to consider R as a right module over itself. By Lemma 2.19 the family $(\{eR\}_{e\in\mathrm{Idem}(R)})$ consists of finitely generated projective right modules. This family is partially ordered by inclusion or, equivalently, by the idempotent relation in Definition 2.7:

$$eR \subseteq fR \iff e \leq f \iff ef = fe = e.$$

The local unit property now ensures that the corresponding union construction exhausts R. □

2.2 Free constructions and path algebras

In this section, we establish free k-algebra constructions starting from a set without any algebraic structure, from a vector space without the multiplicative structure or from a graph. The latter one is implemented by idempotent relations and leads to path algebras.

Definition 2.21. Let X be a set. The *free associative k-algebra* $F(X)$ on X is defined as the *universal* algebra that receives a map from X in the sense that any other map $f \in \mathrm{Map}(X, A)$ into an algebra A factors through $F(X)$ by a unique algebra homomorphism $\varphi_f \in \mathrm{Hom}(F(X), A)$. Phrased diagrammatically:

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & F(X) \arrow[d, dotted, "\exists!\,\mathrm{hom}\,\varphi_f"] \\ & A. \end{tikzcd} \tag{1}$$

Such an algebra actually exists and the universal property determines it up to algebra isomorphism. One convenient construction relies on the vector space which is the formal k-linear span of *words* of finite length $W := \bigcup_{n\in\mathbb{N}} X^n$ with *letters* in the *alphabet* X:

$$F(X) = \mathrm{span}\{x_1 \cdots x_n \mid n \in \mathbb{N},\, x_j \in X \text{ for } j \leq n\} = \bigoplus_{n\in\mathbb{N}} \bigoplus_{(x_j)_{j\leq n}\in X^n} k = \bigoplus_{w\in W} k.$$

The generating set X embeds as the words of length one and the algebra structure is induced by concatenation of words $a \in X^i$ and $b \in X^j$ that are for simplicity written without separators:

$$(a_1 \cdots a_i) \cdot (b_1 \cdots b_j) := a_1 \cdots a_i b_1 \cdots b_j \in X^{i+j}.$$

Now, $F(X)$ admits the desired universal property by extending a map $f \colon X \to A$ into an algebra A first multiplicatively to words $f(x_1 \cdots x_n) := f(x_1) \cdots f(x_n) \in A$ and then linearly to the unique algebra homomorphism

$$\begin{aligned} \varphi_f \colon F(X) &\to A, \\ \sum_{w \in W} \lambda_w w &\mapsto \sum_{w \in W} \lambda_w f(w). \end{aligned}$$

Remark 2.22. Since an element of a free algebra $F(X)$ is a finite linear combination of words, it is characterised by its coefficients $\lambda_w \in k$ that are zero for all but finitely many basis elements indexed by $W = \bigcup_{n \in \mathbb{N}} X^n$. Thus, $F(X)$ can also be understood as the convolution algebra

$$\begin{aligned} F(X) &\cong \left(\mathrm{Maps}_f(W, k), *\right), \\ \sum_{w \in W} \lambda_w w &\mapsto (w \mapsto \lambda_w), \end{aligned}$$

of finitely supported maps with the usual convolution operation

$$\chi_1 * \chi_2(w) := \sum_{u, v \in W \colon uv = w} \chi_1(u)\chi_2(v)$$

and the inclusion $x \in X \mapsto (\delta_x \colon W \to k)$.

Remark 2.23. The unitalisation introduced in Definition 2.12 amounts to adjoining an additional k-summand as the scalar multiples of the new unit element. In the unitalisation $F(X)^+$ of a free algebra, this summand is indexed by the *empty word* as the unique word in $X^0 := \{1\}$ with zero letters, which causes a trivial concatenation of words.

The same pattern of universal properties and free constructions using words in the generating structure applies more generally if one aims to add an additional layer of algebraic structure. It is a purely category theoretical argument. The theme of mapping a set to a free algebra can also be applied to the underlying vector space of an algebra A itself if we forget the already existing multiplication totally or at least partially. Say, except for the prior unit element 1_A if there is any. The common idea is to work with the adjoint pair of forgetful and free functors to obtain a "relatively" free algebra based on the structure we decided to keep. It paves the way for a universal algebra extension that describes A as the quotient of its relatively free algebra with respect to the kept information by its intentionally forgotten relations. See also [15] or [19].

Definition 2.24 ([15, p. 255])**.** Let A be an algebra. The free algebra of its underlying vector space is its *tensor algebra*

$$TA := \bigoplus_{n\geq 1} A^{\otimes n}$$

with the linear inclusion $A \to TA$ as tensors of length one and the canonical multiplication that is induced by tensoring. Analogously to Definition 2.21, it is the universal algebra that admits a linear map from A since for any other linear map $f\colon A \to B$ into an algebra B the factorwise application is the only possibility to extend it multiplicatively on pure tensors. Phrased diagrammatically as in (1):

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "f\,\mathrm{lin}"'] & TA \arrow[d, dashed, "\exists!\,\mathrm{hom}\,\varphi_f"] \\ & B. \end{tikzcd} \tag{2}$$

Definition 2.25 ([15, p. 255])**.** Linear maps between unital algebras that also preserve the multiplicative unit are called *based-linear*. In case of a unital algebra A the tensor algebra TA can be first unitalised in the usual way by adjoining a summand $A^0 = k$ and then adapted to based linear maps by implementing the relation that multiplication by $1_A \in A^1 \subset TA^+$ acts as the the identity on both sides. In this way, we obtain the *based tensor algebra*

$$T_bA := \frac{TA^+}{TA^+(1_A - 1_k)TA^+}$$

on which the inclusion $A \to TA$ is turned into a canonical based linear map

$$\hat{\rho}\colon A \to T_bA$$

by construction. Thus, T_bA is the universal unital algebra that admits a based linear map from A:

$$\begin{tikzcd} A \arrow[r, "\hat{\rho}"] \arrow[dr, "\rho\text{ based lin}"'] & T_b A \arrow[d, dotted, "\exists!\text{ hom }\varphi_\rho"] \\ & B. \end{tikzcd} \tag{3}$$

Definition 2.26. In the setup of (2), the linear map Id_A yields the surjective *multiplication homomorphism* $\mu \colon TA \to A$ that erases all tensors. If we call its kernel JA, this gives rise to the *free algebra extension*

$$JA \hookrightarrow TA \twoheadrightarrow A \tag{4}$$

with the inclusion $\iota \colon A \to TA$ as a linear section. In case of a unital algebra A, the identity is even based-linear and hence yields an analogous free based algebra extension

$$J_b A \hookrightarrow T_b A \twoheadrightarrow A$$

with $\hat{\rho}$ as in Definition 2.25 as based linear section.

Besides sets or vector spaces, one can also construct associated algebras based on other structural information. In principle, for the convolution idea in Remark 2.22 just a semi-group structure is needed. In case of $(W, \cdot)$, it is given by concatenation of words. There are plenty of other possibilities, though.

Example 2.27 ([20, p. 32]). The morphisms in a category $C = (C^0, C^1)$ also admit a semigroup structure given by its composition as long as we additionally introduce a formal morphism 0 that serves as a result for incomposable morphisms in C^1:

$$\begin{aligned} () \cdot () \colon C^1 \cup \{0\} \times C^1 \cup \{0\} &\to C^1 \cup \{0\} \\ f \cdot g &:= \begin{cases} f \circ g, & f, g \in C^1 \text{ composable}, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

Since the composition $f \circ g$ of any two morphisms $f \colon d \to e$ and $g \colon b \to c$ exists if and only if the objects c and d agree, identity morphisms can be used to detect this case. Note that for any two objects $c, d \in C^0$ the identity morphisms satisfy

$$1_d \cdot 1_c = \delta_{c,d} 1_c.$$

As a special case, however, we just focus on categories whose morphisms consist of finite paths on a countable directed graph E. In this setup, the vertices take the role of identity morphisms while the edges generate the rest of the countable morphism set.

Definition 2.28 ([2, Def. 1.2.4])**.** Let k be a field and let

$$E := (E^0,\, E^1,\, s\colon E^1 \to E^0,\, r\colon E^1 \to E^0)$$

be shorthand for a directed graph, also known as a *quiver*. Throughout, it is assumed that both the set of *vertices* E^0 and the set of *edges* E^1 are countable.
Let kE denote its *path algebra*, that is, the free associative k-algebra generated by $E^0 \cup E^1$ subject to the relations:

$$\begin{aligned} &(V) \quad vw = \delta_{v,w} v \quad \forall\, v, w \in E^0, \\ &(E) \quad s(e)e = er(e) = e \quad \forall\, e \in E^1. \end{aligned}$$

The set E^n of reduced words with $n \in \mathbb{N}$ letters in E^1 consists of so called *paths* of length n. Due to the expansion in (E) we observe that any path $p = e_1 \cdots e_n \in E^n$ has *compatible* edges $r(e_i) = s(e_{i+1})$ for $1 \leq i < n$ as well as a well-defined source $s(p) := s(e_1)$ and range $r(p) := r(e_n)$. In this context, a vertex arises as a path of length zero with $s(v) = r(v) := v$.
Note that $\mathrm{Path}(E) := \bigcup_{n\in\mathbb{N}_0} E^n$ forms a basis of kE by design and defines a canonical $\mathbb{N}_0$-*grading* induced by the length of paths:

$$|p| := n \quad \forall\, p \in E^n \subseteq \mathrm{Path}(E).$$

Further note that the algebra multiplication is defined as a k-linear extension of path concatenation from left to right.

Remark 2.29. Depending on the context, it might also be desirable to exchange the roles of s and r in Definition 2.28 in order to have an effective path concatenation from right to left. Since this is standard for the composition $() \circ ()$ of operators or maps in general, many authors and especially operator algebraists are in favor of this inverted convention. However, in this thesis we are going to stick with the “left-to-right” approach as it is often used in a graph theoretical context or in [2] as well.

Example 2.30. Let R_n be the quiver that was already mentioned in the introduction. It consists of one vertex $v = 1$ and n loops $e_j\colon v \to v$ for $j \leq n$. By design, the single vertex serves as a unit, or in other words, as the unique empty path of length 0. Moreover, all finite words in the letters e_j are reduced paths in $\mathrm{Path}(R_n)$. Hence, the corresponding path algebra is nothing but the unitalisation of the free associative algebra in the generators e_j:

$$kR_n = k[e_1, \dots, e_n]^+.$$

In particular, R_1 yields the polynomials in one variable.

Remark 2.31. For a general quiver, the set of vertices E^0 becomes a set of pairwise orthogonal idempotents in kE, while the source and range assignment for paths allow for a decomposition

$$kE = \bigoplus_{v,w \in E^0} vkEw.$$

As a result, kE has enough idempotents since $\mathrm{span}(E^0)$ provides a set of local units as in Definition 2.8.

2.3 Square-zero extensions and quasi-free algebras

Classical geometry and their well-studied structural objects such as varieties or even manifolds can be described via algebras and algebraic tools that turn out to remember all geometric information. For the study of varieties, one might as well investigate quotients of multivariate polynomial algebras, while manifolds correspond to algebras of smooth functions over them. Even sheaves can be described algebraically.

A common theme in mathematics is to explore multiple viewpoints for the same structure. In the 20th century, this procedure is formalised by category theory, where the objects of interest make up the objects of the category at hand and their structure preserving maps serve as the morphisms between them. Two such categories are conceptually the same once there is an equivalence of categories.

With this interpretation we can shift classical geometry to their corresponding commutative algebras. However, the algebraic concepts also work in a noncommutative setup and this motivates to stretch the known correspondence to a formal underlying noncommutative geometry. This idea can be carried out at many points and in vast generality and thus it can get arbitrarily complicated. If we just try to generalise the treatment of a nonsingular, that is, a smooth affine variety, however, we end up with a quite restrictive class of algebras on the other site and the key property is quasi-freeness. This notion is introduced and studied by Cuntz and Quillen in [15]. For an algebra A, it corresponds to the ability to lift any homomorphism $A \to R/I$ into the quotient by a nilpotent or, equivalently, by a square-zero ideal $I \subseteq R$ to a homomorphism $A \to R$. Roughly speaking, smoothness is expressed by the ability to regain a proper homomorphism after a slight tilt of multiplicativity. The strength of quasi-free algebras in our context is given by their compatibility with localisations, which effectively allows to also reduce homological computations to the unlocalised algebra later on.

Definition 2.32 ([24, Def. 16.12])**.** An *algebra extension* $A \cong E/I$ is a short exact sequence of algebras

$$0 \longrightarrow I \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 0\,.$$

If we identify I with its image $i(I) = \ker(p) \subseteq E$, an algebra extension describes A as a quotient algebra of E modulo an ideal $I \subseteq E$. If this is a square-zero ideal $I^2 = 0$, we speak of a *square-zero extension.*

Definition 2.33 ([24, Def. 20.1])**.** An algebra A is called *quasi-free* if every square-zero extension splits by an algebra homomorphism:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] & E \arrow[r, "p"] & A \arrow[l, dotted, bend left] \arrow[r] & 0\,. \end{tikzcd}$$

Remark 2.34. On the level of vector spaces, any algebra extension splits by a linear map. To see this, one can pick a basis for A and use that p is surjective to lift all basis elements. This assignment can be extended linearly and defines a linear section by construction. In case of unital algebras A and E, it is useful to refine this construction to take care of the unit elements 1_A and 1_E. For this, we first extend $\{1_A\}$ to a basis of A and then explicitly pick the preimage $1_E \in p^{-1}(1_A)$ for the section. The multiplicative algebra structure, however, is in general not respected by such a linear section $s\colon A \to E$.

Remark 2.35. Any algebra extension E/I of a free algebra $F(X)$ splits by an algebra homomorphism since the surjective quotient map $p\colon E \to F(X)$ allows to pick preimages $\hat{x} \in p^{-1}(x)$ for every $x \in X$ and this assignment extends uniquely to an algebra homomorphism. In particular, free algebras are also quasi-free.

Theorem 2.36 ([24, Def. 20.12])**.** *Let A be an algebra. Then A is quasi-free if and only if for any square-zero extension $I \hookrightarrow E \twoheadrightarrow Q$, any algebra homomorphism $A \to Q$ lifts to* *an algebra homomorphism $A \to E$. Phrased diagrammatically:*

$$\begin{tikzcd} & & A \arrow[d, "f"] \arrow[dl, dotted, "\exists\,\mathrm{hom}"'] \\ I \arrow[r] & E \arrow[r, "p"] & Q. \end{tikzcd}$$

Proof. Let A be quasi-free. Let $I \hookrightarrow E \twoheadrightarrow Q$ be a square-zero extension and let $f\colon A \to Q$ be an algebra homomorphism. Inspired by a similar setup for projective modules over rings mentioned in Remark 2.15, the idea is to build a commuting diagram of algebra homomorphisms

$$\begin{tikzcd} I \arrow[r, dotted] \arrow[d, equal] & E' \arrow[r, dotted, "p'"] \arrow[d, dotted, "f'"] & A \arrow[l, dotted, bend left, "s"] \arrow[d, "f"] \\ I \arrow[r] & E \arrow[r, "p"] & Q \end{tikzcd}$$

such that $f' \circ s$ provides a lift for f. Explicitly, this is done by using the pullback

$$E' = E \times_{p,f} A := \{(e, a) \in E \times A \mid p(e) = f(a)\}$$

in combination with the maps f' and p' as the restrictions of the coordinate mappings. The pullback makes up the universal commuting square over the legs given by p and f and turns the first row into a square-zero extension of A. Indeed, p' is surjective since for any $a \in A$ surjectivity of p allows to pick a preimage E-component in the fibre of $f(a)$. Moreover, p' has the kernel $\{(e, 0) \in E \times A \mid p(e) = f(0)\} = \{(i, 0) \mid i \in \ker(p) = I\} \cong I$. Since A is quasi-free, the first row splits by an algebra homomorphism $s\colon A \to E'$ that fits into the diagram above because

$$(p \circ f') \circ s(a) = (f \circ p') \circ s(a) = f \circ \mathrm{Id}_A(a) = f(a)$$

for $a \in A$. Therefore, the E-component map $f' \circ s\colon A \to E$ of the section lifts f along p, as desired.

Conversely, we can simply choose Id_A for the homomorphism $A \to Q$ that we would like to lift. $\square$

Remark 2.37. This description of quasi-freeness is analogous to the definition of projective modules in Definition 2.16. Roughly speaking, it demands the projective lifting property for algebras and just along those surjective homomorphisms that have a square-zero kernel.

As the name suggests, quasi-free algebras generalise the family of free algebras. It is worth considering this idea to phrase the generalisation in terms of universal properties.

Lemma 2.38 ([15, p. 5])**.** *Let A be an algebra and let TA be its tensor algebra as in Definition 2.24. If we pass to the quotient $\pi\colon TA \to TA/(JA)^2$, we turn its free algebra extension in (4) into its universal square-zero extension*

$$JA/(JA)^2 \hookrightarrow TA/(JA)^2 \twoheadrightarrow A \tag{5}$$

with a linear section $\overline{\iota} := \pi \circ \iota$. The extension is universal in the sense that any other square-zero extension $I \hookrightarrow E \twoheadrightarrow A$ with a specified linear section $s\colon A \to E$ receives a unique algebra homomorphism $\overline{\phi}\colon TA/(JA)^2 \to E$ from it that makes the following diagram commute:

$$\begin{tikzcd}
JA/(JA)^2 \arrow[r] \arrow[d, dotted] & TA/(JA)^2 \arrow[r, "\overline{\mu}"] \arrow[d, dotted, "\exists!\,\overline{\phi}"'] & A \arrow[l, bend left, "\overline{\iota}"] \arrow[d, equal] \\
I \arrow[r] & E \arrow[r, "p"] & A. \arrow[l, bend left, "s"]
\end{tikzcd}$$

In the same fashion

$$J_bA/(J_bA)^2 \hookrightarrow T_bA/(J_bA)^2 \twoheadrightarrow A$$

is the universal square-zero extension with a based linear section induced by $\hat{\rho}$ if A is unital.

Proof. First of all, the multiplication map μ annihilates $(JA)^2 \subseteq IA$ and thus indeed factors through the quotient $\overline{\mu}\colon TA/(JA)^2 \to A$. Now, let $A = E/I$ be any square-zero extension with a linear section $s\colon A \to E$. By the universal property of TA in (2), the section s yields a unique homomorphism $\phi\colon TA \to E$. Since $(p \circ \phi) \circ \hat{\rho} = p \circ s = \mathrm{Id}_A$, the universal property of TA again yields $p \circ \phi = m$ by uniqueness. As a result, $\phi(JA) \subseteq \ker(p) = I$ and thus ϕ indeed factors through $\overline{\phi}\colon TA/(JA)^2 \to E$ because $\phi(JA^2) = \phi(JA)^2 \subseteq I^2 = 0$. By construction, $p \circ \overline{\phi} = \overline{\mu}$ now shows that the diagram above commutes. The same argument applies in the based linear case with the universal property (3) of T_bA instead. □

Corollary 2.39. *An algebra A is quasi-free if and only if its universal square-zero extension in (5) splits by an algebra homomorphism $q\colon A \to TA/(JA)^2$.*

Proof. In the quasi-free case, the existence of q follows immediately. Conversely, for any square-zero extension $A = E/I$, the construction in Lemma 2.38 shows that $\overline{\phi} \circ q$ provides the desired section since $p \circ (\overline{\phi} \circ q) = (p \circ \overline{\phi}) \circ q = \overline{\mu} \circ q = \mathrm{Id}_A$. □

Having elaborated on universal constructions, we now aim to classify square-zero extensions from a different point of view.

Lemma 2.40 ([24, Def. 16.12])**.** *In a square-zero extension*

$$0 \longrightarrow I \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 0$$

of an algebra A, the ideal $i(I) \cong I$ carries a canonical A-bimodule structure. It is given by $a \cdot x \cdot b := s(a)i(x)s(b) \in I$ for $a, b \in A$, a linear section $s\colon A \to E$ and $x \in I$.

Proof. Without loss of generality, we can identify I with its image in E. If the claim is shown for the proposed left module structure, it follows for the right module structure analogously. First of all, the definition is independent of s since two sections s and s' agree modulo I and $(s(a) - s'(a))x \in I^2 = \{0\}$. Next, the expression is well-defined since $I \subseteq E$ is an ideal. Moreover, $\cdot$ is additive since s is linear. So it is only left to show that $\cdot$ is also compatible with multiplication in A. To see this, note that terms

$$\omega_s(a, b) := s(ab) - s(a)s(b) \tag{6}$$

that express the lack of multiplicativity of s for $a, b \in A$ are in I as they are killed by p. Therefore, for $x \in I$, the claim follows from

$$0 + a \cdot (b \cdot x) = \omega_s(a, b)x + a \cdot (b \cdot x) = (ab) \cdot x. \qquad \square$$

The lemma justifies to speak of a square-zero extension of A by an A-bimodule I. Later in Theorem 2.46 it turns out that the map $\omega_s\colon A \times A \to I$ in (6), which measures the failure of multiplicativity of a linear section s, actually classifies the square-zero extension up to equivalence. To have the correct tools and notions of equivalence for this statement at hand, it is a good idea to interpret ω as a special kind of bilinear function, which is called Hochschild 2-cocycle.
In general, square-zero extensions of algebras by bimodules provide a framework, where the n-linearity picture of Hochschild cohomology shows up naturally for $n \leq 2$ and in the following we elaborate on this point. General homological notions aside from Hochschild cohomology will be treated in more detail later on.

Definition 2.41 ([17, pp. 21-23], [24, Def. 16.17])**.** Let A be an algebra and let M be an A-bimodule. Define the *Hochschild cochain complex* as the cocomplex of multilinear maps

$(\mathrm{Mulin}(A^n, M))_{n\geq 0}$ with the coboundary map

$$\begin{aligned}\delta = \delta_n \colon \mathrm{Mulin}(A^n, M) &\to \mathrm{Mulin}(A^{n+1}, M)\\ \varphi &\mapsto \Big((a_1, \dots, a_{n+1}) \mapsto a_1 \cdot \varphi(a_2, \dots, a_{n+1}) + \sum_{j=1}^{n} (-1)^j \varphi(\dots, a_j a_{j+1}, \dots)\Big)\\ &\qquad + (-1)^{n+1}\varphi(a_1, \dots, a_n) \cdot a_{n+1}.\end{aligned}$$

Independently from this chapter, it is verified later in Lemma 4.9 that $\delta^2 = 0$ indeed holds. The proof becomes more elegant in the context of the bar resolution in Definition 4.8 and in terms of the "face map" notation. See also Lemma 5.5. As a result, we obtain a well-defined *Hochschild cohomology*

$$HH^n(A, M) := \frac{\ker(\delta_n)}{\delta_{n-1}(\mathrm{Mulin}(A^{n-1}, M))}$$

of n-cocycles modulo n-coboundaries.

Remark 2.42 ([17, p. 22]). For $n = 0$, we interpret δ_{-1} as the zero map and a 0-linear map as a choice of an element $m \in M$. Thus, $\delta_0(m)(a) = a \cdot m - m \cdot a$ for $a \in A$ and the zeroth Hochschild cohomology group is nothing but the centre of the bimodule M:

$$HH^0(A, M) = \ker(\delta_0) = M^{\#} := \{m \in M \mid ma = am \quad \forall a \in A\}.$$

Remark 2.43 ([17, p. 23], [20, p. 23]). For $n = 1$, a Hochschild 1-cocycle is a linear map $\gamma \colon A \to M$ that satisfies the *Leibniz rule*

$$\gamma(ab) = a\gamma(b) + \gamma(a)b \quad \forall a, b \in A \iff \delta_\gamma = 0.$$

It is also known as a *derivation* $\gamma \in \mathrm{Der}(A, M)$. A 1-coboundary, however, is of the shape $\delta_0(m)$ that takes the (negative) commutator with the element $m \in M$ as in Remark 2.42. Thus, the first Hochschild cohomology group is the quotient of derivations by the subgroup of *inner derivations* $\mathrm{Inn}(A, M) := \{\mathrm{ad}_x := [x, \cdot] \colon A \to M,\ a \mapsto xa - ax \mid m \in M\}$:

$$HH^1(A, M) = \frac{\mathrm{Der}(A, M)}{\mathrm{Inn}(A, M)}.$$

To get back to extensions of A by M, we now introduce a very canonical one and relate its multiplicative sections to derivations.

Definition 2.44 ([24, p. 82])**.** Let A be an algebra and let M be an A-bimodule. Define the *crossed product* as their direct sum equipped with the untwisted algebra structure

$$\begin{aligned} A \ltimes M = M \rtimes A &:= (M \oplus A, \cdot) \\ (m_1, a_1) \cdot (m_2, a_2) &:= (m_1 a_2 + a_1 m_2, a_1 a_2). \end{aligned}$$

Note that $\cdot$ is indeed associative and that $(0,1)$ is a unit element for $M \rtimes A$ if A is unital. Moreover, the direct sum environment provides a square-zero extension

$$0 \longrightarrow M \xrightarrow{(\mathrm{Id}_M, 0)} M \rtimes A \xrightarrow{(0, \mathrm{Id}_A)} A \longrightarrow 0\,. \tag{7}$$

Theorem 2.45 ([24, p. 82])**.** *Let A be an algebra and let M be an A-bimodule. Then the set of multiplicative sections* $\mathrm{Split}(A, M)$ *of the direct sum extension* (7) *is in natural bijection with the set of derivations* $\mathrm{Der}(A, M)$ *via*

$$\begin{aligned} &\mathrm{Split}(A, M) \cong \mathrm{Der}(A, M), \\ &s = (\gamma, \mathrm{Id}_A) \mapsto \gamma. \end{aligned}$$

The equivalence relation that defines the first Hochschild cohomology group $HH^1(A, M)$ now induces an equivalence relation on $\mathrm{Split}(A, M)$*. In case of a unital A where $1 \in A$ acts as the identity on M, this leads to the notion that two sections $s, s' \in$* $\mathrm{Split}(A, M)$ *are considered equivalent if they are conjugate by an element $(x, 1) \in M \times \{1_A\}$, that is, if they agree up to post-composition with an inner automorphism*

$$\begin{aligned} \mathrm{Ad}_{(x,1)} \colon M \rtimes A &\to M \rtimes A, \\ (m, a) &\mapsto (x, 1) \cdot (m, a) \cdot (x, 1)^{-1}, \end{aligned}$$

for some $x \in M$. Phrased in a diagram, this reads as

$$\begin{tikzcd} M \rtimes A \arrow[d, dotted, "\mathrm{Ad}_{(x,1)}"] & A \arrow[l, "s"'] \arrow[dl, bend left, "s'"] \\ M \rtimes A. & \end{tikzcd}$$

Proof. First of all, linear sections $s \colon A \to M \rtimes A$ are precisely of the shape $s = (\gamma, \mathrm{Id}_A)$ for a linear map $\gamma \colon A \to M$. Now, given $a, b \in A$ the desired bijection immediately follows from the observation

$$\begin{aligned} & s(ab) = s(a)s(b) \\ \iff & (\gamma(ab), ab) = (\gamma(a), a) \cdot (\gamma(b), b) = (\gamma(a)b + a\gamma(b), ab) \\ \iff & \gamma \in \mathrm{Der}(A, M). \end{aligned}$$

If we now inspect the induced equivalence relation

$$s = (\gamma, \mathrm{Id}_A) \sim s' = (\gamma', \mathrm{Id}_A) \iff [\gamma] = [\gamma'] \in HH^1(A, M)$$

on $\mathrm{Split}(A, M)$ for the unital setup, cohomologous derivations $\gamma' = \gamma + \mathrm{ad}_x$ with an inner derivation for $x \in M$ translate to the claimed form:

$$\begin{aligned} (\mathrm{Ad}_{(x,1)} \circ s)(a) &= (x, 1) \cdot (\gamma(a), a) \cdot (x, 1)^{-1} \\ &= (x, 1) \cdot (\gamma(a), a) \cdot (-x, 1) \\ &= (\gamma(a) + xa - ax, a) \\ &= (\gamma(a) + \mathrm{ad}_x(a), a) \\ &= s'(a). \end{aligned}$$

□

Theorem 2.46 ([17, p. 23], [24, Thm. 16.13])**.** *There is a natural bijection between equivalence classes of square-zero extensions of an algebra A by an A-bimodule M and the second Hochschild cohomology group $HH^2(A, M)$. Two square-zero extensions of A by M are considered equivalent if there is an algebra isomorphism making the following diagram commute:*

$$\begin{tikzcd} M \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, dotted, "\cong"] & A \arrow[d, equal] \\ M \arrow[r] & E_2 \arrow[r] & A. \end{tikzcd}$$

Proof. Let $M \hookrightarrow E \twoheadrightarrow A$ be a square-zero extension, let s a linear section for it, and let $\omega_s \colon A \times A \to i(M) \cong M$ be as in (6). The first claim is that ω_s defines a Hochschild 2-cocycle. Indeed, bilinearity of ω_s follows immediately from the linearity of s. Moreover, the induced trilinear map $\delta\omega_s\colon A\times A\times A\to M$ is identically zero:

$$\begin{aligned}\delta\omega_s(a,b,c) &= a\omega_s(b,c)-\omega_s(ab,c)+\omega_s(a,bc)-\omega_s(a,b)c\\ &= (s(a)s(bc)-s(a)s(b)s(c))-(s(abc)-s(ab)s(c))\\ &\quad +(s(abc)-s(a)s(bc))-(s(ab)s(c)-s(a)s(b)s(c))\\ &= 0.\end{aligned}$$

Another linear section differs from s by a linear map $\psi\colon A\to\ker(p)\cong M$. The corresponding 2-cocycles ω_s and $\omega_{s+\psi}$ are cohomologous since their difference is the Hochschild 2-coboundary $\delta\psi$:

$$\begin{aligned}\omega_{s+\psi}(a,b) &= (s+\psi)(ab)-(s+\psi)(a)\cdot(s+\psi)(b)\\ &= s(ab)+\psi(ab)-s(a)s(b)-s(a)\psi(b)-\psi(a)s(b)-\psi(a)\psi(b)\\ &= \omega_s(a,b)+\psi(ab)-a\psi(b)-\psi(a)b.\end{aligned}$$

Moreover, a split linear map for an equivalent extension is given by postcomposition with the isomorphism at the central spot, just as for the inclusion i. As a result, the associated M-valued functions are identical.
To sum up, there is a well-defined assignment

$$[E/M]\mapsto[\omega_s] \tag{8}$$

that maps the equivalence class of a square-zero extension to the cohomology class of the associated 2-cocycle. The next goal is to show injectivity. To see this, note that the linear map $\mathrm{Id}_E-s\circ p$ takes values in $i(M)$ such that s provides explicit vector space isomorphisms

$$(i,s)\colon M\oplus A\to E,\quad (i^{-1}(\mathrm{Id}_E-s\circ p),p)\colon E\to M\oplus A$$

that are inverse to each other. Thus, up to equivalence, the square-zero extension is determined by the induced multiplication map on $M\oplus A$. Concretely, for $a_j\in A$, $m_j\in M$ with $j=1,2$ it is already pinned down by ω_s because

$$\begin{aligned}(i(m_1)+s(a_1))\cdot(i(m_2)+s(a_2)) &= i(m_1)i(m_2)+i(m_1)s(a_2)+s(a_1)i(m_2)+s(a_1)s(a_2)\\ &= 0+i(m_1)s(a_2)+s(a_1)i(m_2)+s(a_1a_2)-\omega_s(a_1,a_2).\end{aligned}$$

Under the usual identification of M with its image $i(M) \subseteq E$, we recognise the A-bimodule action and using the external direct sum notation we read off the multiplication map

$$(m_1, a_1) \cdot_{\omega_s} (m_2, a_2) := (m_1 a_2 + a_1 m_2 - \omega_s(a_1, a_2), a_1 a_2) \tag{9}$$

for the equivalent square-zero extension $M \to (M \oplus A, \cdot_{\omega_s}) \to A$. Thus, the assignment (8) above is injective. In fact, any Hochschild 2-cocycle ω yields an associative multiplication $\cdot_\omega$ that performs a twist by ω in the M-component defined as in (9) because

$$\begin{aligned}
&(m_1, a_1) \cdot_\omega ((m_2, a_2) \cdot_\omega (m_3, a_3)) - ((m_1, a_1) \cdot_\omega (m_2, a_2)) \cdot_\omega (m_3, a_3)\\
&\quad = (m_1, a_1) \cdot_\omega (m_2 a_3 + a_2 m_3 - \omega(a_2, a_3), a_2 a_3)\\
&\qquad - (m_1 a_2 + a_1 m_2 - \omega(a_1, a_2), a_1 a_2) \cdot_\omega (m_3, a_3)\\
&\quad = (m_1 a_2 a_3 + a_1(m_2 a_3 + a_2 m_3 - \omega(a_2, a_3)) - \omega(a_1, a_2 a_3), a_1 a_2 a_3)\\
&\qquad - ((m_1 a_2 + a_1 m_2 - \omega(a_1, a_2))a_3 + a_1 a_2 m_3 - \omega(a_1 a_2, a_3), a_1 a_2 a_3)\\
&\quad = (-a_1 \omega(a_2, a_3) - \omega(a_1, a_2 a_3) + \omega(a_1, a_2) a_3 + \omega(a_1 a_2, a_3), 0)\\
&\quad = (0, 0).
\end{aligned}$$

Hence, (8) is also surjective and a concrete inverse map is given by

$$[\omega] \mapsto [M \to (M \oplus A, \cdot_\omega) \to A],$$

where shifts $\delta\psi$ for a linear $\psi \colon A \to M$ translate into shifts of the canonical section $(0, \mathrm{Id}_A) \colon A \to M \oplus A$ by $(\psi, 0)$. □

Corollary 2.47. *An algebra A is quasi-free if and only if $HH^2(A, M) = 0$ for every A-bimodule M.*

Proof. By Lemma 2.40 every square-zero extension is an extension of A by a bimodule and by Theorem 2.46 such a square-zero extension splits by an algebra homomorphism if and only if the class of associated Hochschild 2-cocycles is trivial. □

To sum up, the Hochschild cohomology groups $HH^j(A, M)$ for $j \leq 2$ have an interpretation in the context of square-zero extensions $M \hookrightarrow E \twoheadrightarrow A$. While the second classifies square-zero extensions up to algebra isomorphisms, the first is concerned with the trivial class of the untwisted extension (7) and determines its sections up to certain inner automorphisms. More concretely, these automorphisms conjugate with square-zero perturbations of the identity $M + \mathrm{Id}_{M \rtimes A} \subseteq (M \rtimes A)^+$. Finally, $HH^0(A, M) = M^\#$ classifies those elements $z \in M$, for which conjugation by $z + \mathrm{Id}_{M \rtimes A}$ is already trivial itself. Of course, in the unital case, we would like to replace $1_{(M \rtimes A)^+} = \mathrm{Id}_{M \rtimes A}$ by the multiplication operator 1_A to speak less technically of an element $(z, 1_A) \in M \rtimes A$ as in Theorem 2.45. However, this needs the unitality assumption that 1_A acts as the identity on M. For our considerations, such a restriction to non-degenerate modules causes no loss of generality as the following lemma shows.

Lemma 2.48 ([24, Lemma 20.3])**.** *Let A be a unital algebra. The following are equivalent:*

1. *A is quasi-free.*
2. *$HH^2(A, M) = 0$ for all non-degenerate bimodules.*
3. *Any square-zero extension $A = E/M$ with unital E splits by an algebra homomorphism.*

Proof. Let M be an A-bimodule. Measure the failure of the unit idempotent $1_A \in A$ to act as the identity on both sides by decomposing M into four direct summands

$$\begin{aligned} M_{11} &:= 1_A \cdot M \cdot 1_A, \\ M_{01} &:= (\mathrm{Id} - 1_A) \cdot M \cdot 1_A, \\ M_{10} &:= 1_A \cdot M \cdot (\mathrm{Id} - 1_A), \\ M_{00} &:= (\mathrm{Id} - 1_A) \cdot M \cdot (\mathrm{Id} - 1_A). \end{aligned}$$

By construction, a 1-index indicates unitality, while a 0-index indicates that any

$$a = 1_A a 1_A = a 1_A = 1_A a \in A$$

acts by zero on the respective side. The next step is to see that all Hochschild 2-cocycles $\omega \colon A \times A \to N$ for a bimodule N with zero multiplication on at least one side are 2-coboundaries. If it is without loss of generality the left side, then ω is the coboundary of the linear map $\psi := -\omega(1_A, \cdot)$ since for $a, b \in A$ we have

$$\begin{aligned}
\delta\psi(a,b) &= a\psi(b) - \psi(ab) + \psi(a)b \\
&= 0 + \omega(1_A, ab) - \omega(1_A, a)b \\
{}_{[1_A \cdot N = 0]} &= [\omega(a,b) - \omega(a,b) + 1_A\omega(a,b)] + \omega(1_A, ab) - \omega(1_A, a)b \\
&= \omega(a,b) - \omega(1_A a, b) + 1_A\omega(a,b) + \omega(1_A, ab) - \omega(1_A, a)b \\
{}_{[\text{sort}]} &= \omega(a,b) + 1_A\omega(a,b) - \omega(1_A a, b) + \omega(1_A, ab) - \omega(1_A, a)b \\
{}_{[\delta_\omega = 0]} &= \omega(a,b).
\end{aligned}$$

Hence, $HH^2(A, M) \cong HH^2(A, M_{11})$, which reduces the problem to non-degenerate modules. This yields the equivalence of (1) and (2).
For the equivalence with (3), note that the proof of Theorem 2.46 translates (2) into the property that every square-zero extension by a non-degenerate bimodule splits by an algebra homomorphism. With this in mind, it remains to show that a square-zero extension $A = {}^{E}\!/_{M}$ has a unital E if and only if M is non-degenerate.
If E is unital, then the quotient map p is unital and thus the induced bimodule structure from Lemma 2.40 on M is non-degenerate as well. Conversely, if M is non-degenerate, then the algebra $E \cong (M \oplus A, \cdot_\omega)$ as in (9) should be unital, where ω represents its class in $HH^2(A, M)$. Indeed, a quick calculation with $a \in A$ and $m \in M$ shows that $(\omega(1_A, 1_A), 1_A)$ does the job:

$$\begin{aligned}
(m, a) \cdot_\omega (\omega(1_A, 1_A), 1_A) &= (m \cdot 1_A + a \cdot \omega(1_A, 1_A) - \omega(a, 1_A), a \cdot 1_A) \\
&= (m + a \cdot \omega(1_A, 1_A) \mp \omega(a, 1_A) - \omega(a, 1_A), a) \\
{}_{[\delta_\omega = 0]} &= (m, a) \\
&= (m + \omega(1_A, 1_A) \cdot a \mp \omega(1_A, a) - \omega(1_A, a), a) \\
&= (1_A \cdot m + \omega(1_A, 1_A) \cdot a - \omega(1_A, a), 1_A \cdot a) \\
&= (\omega(1_A, 1_A), 1_A) \cdot_\omega (m, a).
\end{aligned}$$

□

Corollary 2.49 ([24, Prop. 20.4]). *The base field k is a quasi-free k-algebra. Moreover, in a square-zero extension $A = {}^{E}\!/_{M}$ of k-algebras any idempotent $a \in A$ can be lifted to an idempotent $\hat{a} \in E$.*

Proof. Since k is unital, it suffices to look at square-zero extensions $k = E/M$ with unital $(E, 1_E)$ by Lemma 2.48. In this setup, however, $\lambda \in k \mapsto \lambda \cdot 1_E \in E$ immediately defines the desired split algebra homomorphism. Thus, k is quasi-free. If we now consider a general square-zero extension $A = {}^{E}\!/_{M}$ and an idempotent $a \in A$, then it restricts to a square-zero extension

$$M \to p^{-1}(\operatorname{span}\{a\}) \to \operatorname{span}\{a\}.$$

of the subalgebra $\operatorname{span}\{a\} \cong k$ because of $M = p^{-1}(0) \subseteq p^{-1}(\operatorname{span}\{a\})$.
This extension, however, splits by an algebra homomorphism $s\colon \operatorname{span}\{a\} \to p^{-1}(\operatorname{span}\{a\})$ since k is quasi-free. The lifted element $\hat{a} := s(a) \in p^{-1}(\operatorname{span}\{a\}) \subset E$ is again an idempotent because $\hat{a}^2 = s(a)^2 = s(a^2) = s(a) = \hat{a}$. □

The possibility to lift idempotents in square-zero extensions turns out to be a fruitful tool for further investigations of quasi-free algebras. In fact, it leads us to the first major non-trivial family of quasi-free algebras besides the in some sense obvious free algebras.

Theorem 2.50 ([24, Thm. 20.8]). *Let $Q = (Q^0, Q^1, s, r)$ be a quiver. Then its path algebra kQ is quasi-free.*

Proof. Let $p\colon E \twoheadrightarrow kQ$ describe a square-zero extension $kQ \cong {}^{E}\!/_{\ker(p)}$. By the universal property of kQ, a homomorphism $\widehat{(.)}\colon kQ \to E$ is equivalent to a family of orthogonal idempotents $(\hat{v})_{v\in Q^0}$ in E and compatible choices $\left(\hat{e} \in \widehat{s(e)}E\widehat{r(e)}\right)_{e\in E^1}$. Thus, the construction of a multiplicative section boils down to the search of preimages of vertices and edges under p with these properties.
When it comes to vertices, we can always lift an idempotent to an idempotent in a square-zero extension by Corollary 2.49. However, we have to be careful to maintain orthogonality, too. Since the set of vertices Q^0 is countable, we may enumerate it as $\{v_i \mid i \in \mathbb{N}\}$, say. This artificial order allows to lift these orthogonal idempotents in a controlled way with some analogy to the Gram-Schmidt procedure that iteratively constructs an orthogonal basis. For the first vertex v_1, any idempotent $\hat{v}_1$ in its fibre obtained by Corollary 2.49 works out. If the first n vertices are already lifted to orthogonal idempotents $\hat{v}_j$ for $j \leq n$, the next vertex lift of v_{n+1} has to be arranged in the orthogonal complement of the idempotent $q_{n+1} := \sum_{j\leq n} \hat{v}_j$ that reflects all prior choices. Let $E_{n+1} := \{e \in E \mid q_{n+1}e = 0 = eq_{n+1}\}$ and note that any $e \in E$ can be modified by its one-sided and two-sided products with

q_{n+1} in the sense that $e - q_{n+1}e - eq_{n+1} + q_{n+1}eq_{n+1}$ now belongs to E_{n+1}:

$$\begin{aligned} q_{n+1}(e - q_{n+1}e - eq_{n+1} + q_{n+1}eq_{n+1}) &= q_{n+1}e - q_{n+1}^2 e - q_{n+1}eq_{n+1} + q_{n+1}^2 eq_{n+1} \\ &= 0 \\ &= eq_{n+1} - q_{n+1}eq_{n+1} - eq_{n+1}^2 + q_{n+1}eq_{n+1}^2 \\ &= (e - q_{n+1}e - eq_{n+1} + q_{n+1}eq_{n+1})q_{n+1}. \end{aligned}$$

If we apply this for any preimage $\tilde{v}$ of v_{n+1}, the orthogonality down in kQ guarantees that its modified expression still lifts v_{n+1}:

$$\begin{aligned} p(\tilde{v} - q_{n+1}\tilde{v} - \tilde{v}q_{n+1} + q_{n+1}\tilde{v}q_{n+1}) &= v_{n+1} - \sum_{j \le n} v_j v_{n+1} - v_{n+1} \sum_{j \le n} v_j + \sum_{i,j \le n} v_i v_{n+1} v_j \\ &= v_{n+1} \in p(E_{n+1}). \end{aligned}$$

Therefore, the given extension can be restricted to a square-zero extension

$$0 \longrightarrow I \longrightarrow E_{n+1} \longrightarrow \operatorname{span}\{v_{n+1}\} \longrightarrow 0\,.$$

Now, Corollary 2.49 finally gives an idempotent $\hat{v}_{n+1} \in E_{n+1}$ that lifts v_{n+1} and that is orthogonal to all $\hat{v}_j$ for $j \le n$ by construction. Hence, we have found a way to lift a countable family of orthogonal idempotents in a square-zero extension.
The treatment of the edges now basically comes for free. Let $e \in Q^1$ be an edge and denote its already lifted source and range vertices by $\hat{s}(e)$ and $\hat{r}(e)$ for notational convenience. Since the map p is both multiplicative and surjective, we have

$$p(\hat{s}(e)E\hat{r}(e)) = s(e)p(E)r(e) = s(e)kQr(e).$$

Hence, there are lifts in $\hat{s}(e)E\hat{r}(e) \cap p^{-1}(e)$ and every element $\hat{e}$ in it meets the conditions in the construction of the multiplicative section. □

Lemma 2.51 ([24, Prop. 20.9]). *Let A be a unital algebra and let $A \cong {}^{E}\!/\!_{I}$ be a unital square-zero extension with the usual notation. Then any $\hat{a} \in E$ lifting an element $a \in A$ that is invertible from at least one side in A has the same invertibility behaviour in E.*

Proof. Let $a \in A$ be an element that has a right inverse $b \in A$ with $ab = 1_A$. If we pick preimages $e \in p^{-1}(a)$ and $f \in p^{-1}(b)$, then ef and 1_E both lift $ab = 1_A$, so their difference is in I and has a zero square. Spelled out, this means that

$$(1_E - ef)^2 = 1_E - 2ef + efef = 0 \iff 1_E = (2-ef)ef = ef(2-ef) \iff (ef)^{-1} = 2-ef.$$

In particular, e has a right inverse while f has a left inverse. This shows the claim for one-sided invertibility.
If we now assume two-sided invertibility $ab = 1_A = ba$, the above equivalences are still valid and also copy for exchanged roles of a and b. That is, $(ef)^{-1} = 2 - ef$ and $(fe)^{-1} = 2 - fe$. In particular, e and f both have left and right inverses. Hence, both are invertible with $e^{-1} = 2f - fef$ and $f^{-1} = 2e - efe$, as claimed. □

This observation that lifts in a square-zero extension maintain the invertibility properties of the lifted element relies on an idea that still works out in a nonunital framework for lifted matrices that are at least one-sided invariant under multiplication by an idempotent.

Proposition 2.52. *Let A be an algebra and let ${}^{E}\!/_{I}$ be a square-zero extension of A.*

1. *The quotient map $\pi\colon E \to A$ provides an algebra homomorphism*

   $$\pi_*\colon M_n(E) \to M_n(A)$$

   for all $n \in \mathbb{N}$ that applies π entrywise. This yields a square-zero extension

   $$M_n(I) \longrightarrow M_n(E) \xrightarrow{\pi_*} M_n(A)\ .$$

2. *For any idempotent $q \in \mathrm{Idem}(M_n(A))$ with an idempotent lift $\hat{q} \in \mathrm{Idem}(M_n(E))$ the corners of $\hat{q}$ surject onto the respective corners of q. In particular, any matrix $a \in M_n(A)$ in a corner can be lifted to an $\hat{a} \in M_n(E)$ with the same invariance behaviour under $\hat{q}$.*

3. *Let $(a, \hat{a})$ be such a pair for an element $a \in qM_n(A)$ that additionally admits a right inverse in the sense that the equation $ab = q$ is solvable for some $b \in M_n(A)$. Then the lifted equation $\hat{a}\beta = \hat{q}$ is also solvable for some lift $\beta \in \pi_*^{-1}(b)$. Likewise for the equation $ba = q$ if a is in the other corner $M_n(A)q$ instead.*

4. *In the setup of two idempotents $q, p \in \mathrm{Idem}(M_n(A))$ with idempotent lifts $\hat{q}$ and $\hat{p}$, two-sided invertibility of $a \in qM_n(A)p$ in terms of the existence of $b \in pM_n(A)q$ such*

that

$$ab = q,\ ba = p$$

can be maintained in the sense that for any lift $\hat{a} \in \hat{q}M_n(E)\hat{p}$ *there is a unique* $\beta \in \hat{p}M_n(E)\hat{q}$ *with*

$$\hat{a}\beta = \hat{q},\ \beta\hat{a} = \hat{p}.$$

Proof. In the setup of (1), linearity and surjectivity of π immediately pass on to π_* and by definition of matrix multiplication for $x, y \in M_n(E)$ we have

$$\pi_*(xy) = \left(\pi\left(\sum_{j\le n} x_{ij}y_{jk}\right)\right)_{i,k\le n} = \left(\sum_{j\le n} \pi(x_{ij})\pi(y_{jk})\right)_{i,k\le n} = \pi_*(x)\pi_*(y).$$

Furthermore, $\ker(\pi_*) = M_n(I)$ follows by definition and it has a vanishing square since matrix multiplication causes sums of I^2 elements in every entry. This proves (1).
It is possible to find an idempotent lift $\hat{q} \in M_n(E)$ for the idempotent q by Corollary 2.49 and as in the proof of Theorem 2.50 we have

$$\pi_*(\hat{q}M_n(E)) = \pi_*(\hat{q})\pi_*(M_n(E)) = qM_n(A)$$

for left and analogously for right corners. In particular, any $a \in qM_n(A)$ admits a lift $\hat{a} \in \hat{q}M_n(E)$, as claimed in (2).
For such a pair $(a, \hat{a})$, the equation $ab = q$ can be treated exactly as in Lemma 2.51: If $\hat{b}$ is any candidate lift for b, both $\hat{a}\hat{b}$ and $\hat{q}$ lift q. Therefore, their difference lifts the zero matrix and has to be a matrix with vanishing square. Opening the square gives

$$0 = (\hat{a}\hat{b} - \hat{q})^2 = \hat{a}\hat{b}\hat{a}\hat{b} - \hat{q}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{q} + \hat{q}^2 \iff \hat{q} = \hat{a}(\hat{b} + \hat{b}\hat{q} - \hat{b}\hat{a}\hat{b}).$$

In particular, $\beta_r := \hat{b} + \hat{b}\hat{q} - \hat{b}\hat{a}\hat{b} \in \pi_*^{-1}(b)$ serves as an explicit solution of the equation $\hat{a}\beta = \hat{q}$, which shows (3). Note, however, that if we use the same notation for the corresponding argument for the corner $M_n(A)q$ and the equation $ba = q$, we end up with the slightly different solution

$$\beta_l := \hat{b} + \hat{q}\hat{b} - \hat{b}\hat{a}\hat{b} \in \pi_*^{-1}(b).$$

Finally, in the setup of (4), both one-sided applications of (3) yield β_l and β_r, which are the same unique two-sided inverse:

$$\beta_l = \beta_l \hat{q} = \beta_l(\hat{\alpha}\beta_r) = (\beta_l\hat{\alpha})\beta_r = \hat{p}\beta_r = \beta_r. \qquad \square$$

To sum up, quasi-freeness holds for free algebras and is known to be compatible with at least some relations that one could additionally impose on generators. Among them are idempotent relations, even countably many orthogonal idempotent relations, corner memberships, and generalised invertibility relations that turn out to correspond to the concept of Cohn localisations.

2.4 Cohn localisations

In principle, there are two ways to realise a specific mathematical structure. On the one hand, one can implement the desired properties by imposing relations on a free construction as in the case of path algebras. On the other hand, however, one can also abstractly demand a universal property to define a certain structure as long as the existence of the desired object can be ensured as in the case of tensor products.
In the same fashion, one can arrive at Leavitt path algebras in two different ways: Either by imposing Cuntz-Krieger relations on the extended path algebra or by demanding that certain joint prolongation maps become invertible in a somehow enhanced path algebra. To make this second approach precise, we need the concept of Cohn localisations.

Definition 2.53 ([23, Def. 3.6.11], [3, Def. 2.12]). Let R be a ring with local units and let $(u_i\colon P_R^{(i)} \to Q_R^{(i)})_{i\in I}$ be a set of R-module homomorphisms between projective and finitely generated R-modules. For simplicity, the following definition is carried out for right R-modules although it also works for left R-modules along the same lines.
The *Cohn localisation* of R at the maps $(u_i)_{i\in I}$ is the universal ring $\mathrm{Cohn}(R)$ with a homomorphism $R \to \mathrm{Cohn}(R)$ such that the induced maps

$$u_i \otimes_R \mathrm{Cohn}(R)\colon P^{(i)} \otimes_R \mathrm{Cohn}(R) \to Q^{(i)} \otimes_R \mathrm{Cohn}(R)$$

are one-sided $\mathrm{Cohn}(R)$-module isomorphisms for all $i \in I$. In this context, the adjective "universal" is meant to be in the sense that if there is any other ring D with a homomorphism $f\colon R \to D$ such that the induced maps $u_i \otimes_R D$ are left D-module isomorphisms for all $i \in I$, then f factors uniquely through $\mathrm{Cohn}(R)$:

$$\begin{tikzcd} R \arrow[r] \arrow[dr, "f"'] & \mathrm{Cohn}(R) \arrow[d, dotted, "\exists!"] \\ & D. \end{tikzcd} \tag{10}$$

If $R = A$ is even a k-algebra with local units, then, just as for the concept of unitalisation in Definition 2.12, the Cohn localisation is replaced by the k-linearised version of the ring theoretical construction. In this case, we refer to the universal algebra $\mathrm{Cohn}(A)$ with an algebra homomorphism $A \to \mathrm{Cohn}(A)$ such that the induced maps $u_i \otimes_A \mathrm{Cohn}(A)$ are module isomorphisms for all $i \in I$ as its Cohn localisation instead.

In some sense, finitely generated and projective modules are just slightly more general than free modules. To get an idea of the effect of a Cohn localisation, note that a module homomorphism between free R-modules can be encoded by a matrix and that the localisation at this matrix corresponds to the ability to invert it over the localised ring. At first glance, this concept seems to be different from classical localisation in commutative ring theory. However, the aim of making certain ring elements invertible is nothing but the Cohn localisation at the corresponding multiplication endomorphisms or (1×1)-matrices, respectively.

Lemma 2.54 ([23, Lemma 3.6.12], [3, Prop. 2.13])**.** *The Cohn localisation of R at $(u_i)_{i\in I}$ in the setup of Definition 2.53 exists and is unique up to a unique ring isomorphism. In case of the algebra construction for $R = A$, the uniqueness statement holds for algebra isomorphisms instead.*

Proof. Any finitely generated projective module is isomorphic to a direct summand of a free module of finite rank. Since R has local units by assumption, Lemma 2.19 yields that such an R-module takes the shape of a corner in R^∞. Hence, we can find for any $i \in I$ a number $n_i \in \mathbb{N}$ and corresponding idempotent matrices $p_i, q_i \in M_{n_i}(R)$ such that $P^{(i)} \cong p_i R^{n_i}$ and $Q^{(i)} \cong q_i R^{n_i}$. Translated to this form, u_i is now equivalent to left matrix multiplication by some $m_i \in q_i M_{n_i}(R) p_i$ by Lemma 2.11. The whole task can now be rephrased in terms of matrix relations. Let S be the ring obtained from R by adjoining entries of a family of $(n \times n)$-matrices $(m_i^*)_{i\in I}$ that satisfy the following equations:

$$p_i m_i^* = m_i^* = m_i^* q_i, \quad m_i^* m_i = p_i, \quad m_i m_i^* = q_i.$$

This means that multiplication by these newly created matrices is a map $m_i^* \cdot (): q_i S^{n_i} \to p_i S^{n_i}$, which turns out to be inverse to matrix multiplication $m_i \cdot (): p_i S^{n_i} \to q_i S^{n_i}$ over the enhanced ring S. The first relation is equivalent to $m_i^* \in p_i M_n(S) q_i$ and ensures that the multiplication map is well-defined with the correct domain and range. The other two relations say that these maps are inverses of each other since multiplication by both idempotents p_i and q_i act as the identity on $p_i S^{n_i}$ and $q_i S^{n_i}$, respectively. As a result, the enlargement $R \to S$ leads to invertible horizontal maps

$$\begin{tikzcd}
P^{(i)} \otimes_R S \arrow[r, "u_i \otimes_R S"'] \arrow[d, "\cong"'] & Q^{(i)} \otimes_R S \arrow[d, "\cong"'] \\
p_i R^{n_i} \otimes_R S \arrow[r, "m_i \cdot () \otimes_R S"'] \arrow[d, "\cong"'] & q_i R^{n_i} \otimes_R S \arrow[d, "\cong"'] \\
p_i S^{n_i} \arrow[r, "m_i \cdot ()"'] & q_i S^{n_i}
\end{tikzcd}$$

for all $i \in I$. If D is any other ring with a homomorphism $f: R \to D$ such that $u_i \otimes_R D$ is invertible for all $i \in I$, then f induces a homomorphism between matrix algebras f_* that transports $m_i \in q_i M_{n_i}(R) p_i$ entrywise to an invertible multiplication map by the matrix $f_*(m_i) \in f_*(q_i) M_{n_i}(D) f_*(p_i)$. Its inverse can again be written as a multiplication map by $f_*(m_i)^* \in f_*(p_i) M_{n_i}(D) f_*(q_i)$.
In order to extend f from R to S, we need to assign elements in D for every adjoined entry in m_i^*. The only possibility to do so is to map m_i^* entrywise to $f_*(m_i)^*$. This defines a unique homomorphism $S \to D$ extending f. Consequently, S admits the desired universal property and as a universal object it is unique up to isomorphism. By abuse of notation, we can therefore refer to the isomorphism class of S as $\mathrm{Cohn}(R)$.
Note that the construction for an algebra $R = A$ goes through along the same lines if the word "ring" is replaced by "algebra" and all mentioned homomorphisms are required to be algebra homomorphisms. $\square$

So, in practice, Cohn localisations are obtained from the unlocalised ring by adjoining new ring elements subject to relations that encode the invertibility of the localisation maps in the end. For our purposes, they are only applied to k-algebras A with local units. The key observation in this section is that the process of Cohn localisation preserves quasi-freeness.

Theorem 2.55. *Let A be a quasi-free algebra with local units. Then any Cohn localisation* *of A remains quasi-free.*

Proof. Let $\mathrm{Cohn}(A)$ be the Cohn localisation of A at a family of module homomorphisms $(u_i)_{i\in I}$ between finitely generated and projective A-modules. They correspond to matrix multiplication maps by certain $m_i \in q_i M_{n_i}(A) p_i$ for idempotents p_i, q_i and $i \in I$. By Lemma 2.54, these matrices are invertible over $\mathrm{Cohn}(A)$ in the sense that there are $m_i^* \in p_i M_{n_i}(\mathrm{Cohn}(A)) q_i$ for $i \in I$ whose products with m_i give the idempotents p_i and q_i.
Now, let $I \hookrightarrow E \twoheadrightarrow \mathrm{Cohn}(A)$ be a square-zero extension. Since A is quasi-free, the inclusion into $\mathrm{Cohn}(A)$ lifts to an algebra homomorphism $\varphi\colon A \to E$ by Theorem 2.36. This gives an explicit lift for every m_i. Since an algebra homomorphism preserves idempotents, these lifts are in corresponding corners of the matrix algebra over E:

$$\hat{p}_i := \varphi_*(p_i),\ \hat{q}_i := \varphi_*(q_i),\ \hat{m}_i := \varphi_*(m_i) \in \hat{q}_i M_{n_i}(E) \hat{p}_i.$$

The key observation at this point is that Proposition 2.52 (4) now applies to $a := m_i, \hat{a} := \hat{m}_i$ and $b := m_i^*$. We are able to find a unique solution β in the fibre of m_i^* that satisfies the inverse equations for $\hat{m}_i$ in (4). As a result, the multiplication by $\hat{m}_i$ is invertible, which means that φ induces invertible homomorphisms $E \otimes_A u_i$. Thus, φ uniquely factors through $\mathrm{Cohn}(A)$ by its universal property:

$$\begin{tikzcd}
 & & A \arrow[d] \arrow[dl, dotted, "\varphi"'] \\
I \arrow[r] & E \arrow[r] & \mathrm{Cohn}(A). \arrow[l, dotted, bend left, "\exists\,\mathrm{hom}\, s"]
\end{tikzcd}$$

In other words, we have extended φ to a multiplicative section $s\colon \mathrm{Cohn}(A) \to E$ because it lifts the entries of m_i^* to the corresponding entries of β. This proves quasi-freeness. $\square$

3 Leavitt path algebras as Cohn localisations

3.1 Relative Leavitt path algebras

As a foundation, we deal with path algebras as a possibility to turn a quiver into an algebra. Vertices play a central role in the analysis of path algebras and it is useful to classify them with respect to their fibre under the source map s.

Definition 3.1 ([2, Def. 1.2.2])**.** Let E be a quiver. A vertex $v \in E^0$

- with $0 < |s^{-1}(v)| < \infty$ is called *regular* (otherwise *singular*). The set of regular vertices is called $\mathrm{Reg}(E)$.
- with $s^{-1}(v) = \emptyset$ is a *sink* and the set of sinks is called $\mathrm{Sink}(E)$.
- with $|s^{-1}(v)| = \infty$ is an *infinite emitter* and the set of infinite emitters is called $\mathrm{Inf}(E)$.

If $\mathrm{Inf}(E) = \emptyset$, the quiver E is called *row-finite*.
If $\mathrm{Reg}(E) = E^0$, the quiver E is called *regular*.

Given a nontrivial path in kE that ends at a certain vertex $v \in E^0$, its fibre $s^{-1}(v)$ describes all possibilities to extend the path in a nontrivial way by an additional edge. If we want to remember all of them, we arrive at the joint multiplication map

$$\varphi^v := (\cdot e)_{e \in s^{-1}(v)} \colon kEv \to \prod_{e \in s^{-1}(v)} kEr(e). \tag{11}$$

Now the distinction in Definition 3.1 comes in handy. A sink $v \in \mathrm{Sink}(E)$ leads to the zero map $\varphi^v = 0$ while an infinite emitter provides infinitely many possible prolongations and thus an infinite index set. In case of a regular vertex $v \in \mathrm{Reg}(E)$, however, there are at least one but only finitely many component maps to remember, which allows to identify the direct product with the corresponding direct sum. For this, we use the interpretation of the external direct sum as the direct product with finitely supported elements:

$$\bigoplus_{e \in s^{-1}(v)} kEr(e) = \left\{ (\gamma_e)_e \in \prod_{e \in s^{-1}(v)} kEr(e) \,\middle|\, \gamma_e = 0 \,\text{for all but finitely many}\, e \in s^{-1}(v) \right\}.$$

In particular, the left kE-module homomorphism φ^v can be interpreted as a multiplication map by a matrix from the right. Hence, it qualifies for a Cohn localisation of the path algebra kE, which has the required local units.

Definition 3.2 ([3, Def. 2.12])**.** Let kE be the path algebra of a quiver E over a field k. Let $X \subseteq \operatorname{Reg}(E)$ be a selected set of regular vertices in E. Then $\operatorname{Cohn}^X(E)$ refers to the Cohn localisation of kE at (11):

$$\left(\varphi^v\colon kEv \to \bigoplus_{e\in s^{-1}(v)} kEr(e)\right)_{v\in X}.$$

Lemma 3.3. *The algebra* $\operatorname{Cohn}^X(E)$ *in the setup of Definition 3.2 is obtained by adjoining the set* $\{e^* \mid e \in s^{-1}(X)\}$ *subject to the relations*

$$r(e)e^* = e^* = e^*s(e) \qquad \forall\, e \in s^{-1}(X), \qquad (w.E^*)$$

$$e^*f = \delta_{e,f}r(f) \qquad \forall\, e \in s^{-1}(X),\ f \in E^1, \qquad (w.CK1)$$

$$\sum_{e\in s^{-1}(v)} ee^* = v \qquad \forall\, v \in X. \qquad (CK2)$$

Proof. Following the construction of $\operatorname{Cohn}^X(E)$ in Lemma 2.54, we observe that it is obtained by adjoining entries of matrices that satisfy certain relations. Since kE has enough idempotents, one might use the natural smoothness isomorphism $\operatorname{Cohn}^X(E) \otimes_{kE} kE \cong \operatorname{Cohn}^X(E)$ to reexpress the invertible homomorphisms

$$\operatorname{Cohn}^X(E) \otimes_{kE} \varphi^v\colon \operatorname{Cohn}^X(E) \otimes_{kE} kEv \to \operatorname{Cohn}^X(E) \otimes_{kE} \left(\bigoplus_{e\in s^{-1}(v)} kEr(e)\right)$$

as extensions Φ^v of φ^v for every $v \in X$:

$$\Phi^v\colon \operatorname{Cohn}^X(E)v \to \bigoplus_{e\in s^{-1}(v)} \operatorname{Cohn}^X(E)r(e).$$

In comparison to Lemma 2.54, the construction process now uses the left module environment with the choices

$$I := X, \quad n_v := |s^{-1}(v)|, \quad p_i := v \in M_1(kE) \subseteq M_{n_v}(kE), \quad q_i := \operatorname{diag}(r(e) \mid e \in s^{-1}(v)),$$
$$\text{and} \quad m_v := (e)_{e\in s^{-1}(v)} \in M_{1,n_v}(kE) \subseteq M_{n_v}(kE) \quad \text{for} \quad v \in X.$$

The simple shape of Φ^v even allows to state its postulated inverse Ψ^v outside of the matrical syntax as a sum of homomorphisms

$$\Psi^v = \sum_{e\in s^{-1}(v)} \Psi^v_e\colon \bigoplus_{e\in s^{-1}(v)} \mathrm{Cohn}^X(E)r(e) \to \mathrm{Cohn}^X(E)v,$$

starting from their respective module summands, and each of them is forced to be a right multiplication map by the image of the range vertex in question:

$$e^* := \Psi^v_e(r(e)) = r(e)\Psi^v_e(r(e)) \in r(e)\mathrm{Cohn}^X(E)s(e).$$

Hence, $\mathrm{Cohn}^X(E)$ is explicitly obtained from kE by adjoining certain elements e^* for all $e \in s^{-1}(X)$ that ensure $\Psi^v = \sum_{e\in s^{-1}(v)}() \cdot e^* = (\Phi^v)^{-1}$ for all $v \in X$. With this point of view, well-definedness of the mapping Ψ^v is reflected by the relation $(w.E^*)$ while the other two listed relations turn out to establish Ψ^v as the two-sided inverse of Φ^v. Indeed, Ψ^v is a left inverse if and only if the matrix multiplication by the (1×1)-matrix $(\sum_{e\in s^{-1}(v)} ee^*) \in M_1(\mathrm{Cohn}^X(E))$ acts trivially on $\mathrm{Cohn}^X(E)v$. This happens if and only if this element is v itself and gives the $(CK2)$ relation:

$$\Psi^v \circ (\cdot e)_{e\in s^{-1}(v)} = \mathrm{Id}_{\mathrm{Cohn}^X(E)v} \iff \sum_{e\in s^{-1}(v)} ee^* = v \quad \forall\, v \in X.$$

On the other hand, Ψ^v is a right inverse if and only if the matrix multiplication by the $(|s^{-1}(v)| \times |s^{-1}(v)|)$-matrix $(e^*f)_{e,f\in s^{-1}(v)} \in M_{|s^{-1}(v)|}(\mathrm{Cohn}^X(E))$ acts trivially on

$$\bigoplus_{e\in s^{-1}(v)} \mathrm{Cohn}^X(E)r(e).$$

It happens if and only if it is diagonal with their summandwise units $r(e)$. This almost gives the desired $(w.CK1)$ relation:

$$(\cdot e)_{e\in s^{-1}(v)} \circ \Psi^v = \mathrm{Id}_{\bigoplus_{e\in s^{-1}(v)} \mathrm{Cohn}^X(E)r(e)} \iff e^*f = r(f)\delta_{e,f} \quad \forall\, e, f \in s^{-1}(v).$$

Actually, it is equivalent to $(w.CK1)$ since the statement for the by now unconsidered edges $f \in E^1 \setminus s^{-1}(v)$ already follows from incompatibility by using (E), $(w.E^*)$ and finally $s(e) \perp s(f)$:

$$e^*f = e^*s(e)f = e^*(s(e)s(f))f = 0. \qquad \square$$

Both relations $(w.E^*)$ and $(w.CK1)$ are weak versions of relations that are used to

construct Leavitt path algebras later on since they are only imposed for some and not for all edges $e \in E^1$.
Viewed on its own, the relation $(w.E^*)$ in Lemma 3.3 shows that the adjoined elements e^* also behave like edges in a path algebra but with the reversed orientation compared to the original edge e. In principle, this interpretation can be applied to all edges, which extends the relation to (E^*). This motivates the notion of an extended quiver that already implements such edges before we pass to an algebra.

Definition 3.4 ([2, Def. 1.2.4, Def. 2.1.3]). Let E be a quiver. Its *extended quiver* $\hat{E} = (E^0, E^1 \cup (E^1)^*, \hat{s}, \hat{r})$ is defined as an extension of E with $\hat{r}|_{E^1} = r$ and $\hat{s}|_{E^1} = s$ by appending a formal copy of edges $(E^1)^* = \{e^* \mid e \in E^1\}$, called *ghost edges*, with inverted source and range assignment for all $e \in E^1$:

$$\hat{r}(e^*) := s(e), \quad \hat{s}(e^*) := r(e).$$

Its path algebra $k\hat{E}$ can be interpreted as kE with additional generators $(e^*)_{e\in E^1}$ subject to

$$r(e)e^* = e^* = e^*s(e). \tag{E^*}$$

If we put a negative weight $\deg(e^*) := -1$ on these ghost edges, this extends the $\mathbb{N}_0$-grading on kE in Definition 2.28 to a $\mathbb{Z}$-grading on $k\hat{E}$. Concretely, for $m \in \mathbb{Z}$, the assignments $\deg(e) = |e| = 1$ and $\deg(v) = |v| = 0$ for $v \in E^0$ allow to define the subspace of *m-homogeneous* elements as

$$(k\hat{E})_m := \operatorname{span}_k\left\{x_1 \cdots x_n \in \operatorname{Path}(\hat{E}) \;\middle|\; \sum_{j\leq n} \deg(x_j) = m\right\}.$$

With an eye on $*$-algebras, it is convenient to consider ghost edges e^* for all edges and not just for those that start in our specific regular set of vertices X because this allows to define a canonical $*$-involution on $k\hat{E}$ based on the suggestive notation for ghost edges.

Definition 3.5 ([2, Def. 2.0.8]). Let E be a quiver and let $k\hat{E}$ be the path algebra of the extended quiver over the field k. Then an involutive automorphism $\overline{(.)}$ for k, such as complex conjugation for $k = \mathbb{C}$, induces a canonical *-involution on $k\hat{E}$ given by

$$\begin{aligned} &v \in E^0 \mapsto v, \\ &\gamma = e_1 \cdots e_n \in E^n \mapsto \gamma^* := e_n^* \cdots e_1^* \quad \forall\, n \in \mathbb{N}, \\ &\sum_j \lambda_j \gamma_j \mapsto \sum_j \overline{\lambda}_j \gamma_j^* \quad \forall\, \lambda_j \in k,\ \gamma_j \in \mathrm{Path}(E). \end{aligned}$$

Note that we have $(k\hat{E})_m^* = (k\hat{E})_{-m}$ for all $m \in \mathbb{Z}$ by design.

The advantage of $k\hat{E}$ is that it already contains all generators that are needed in order to conveniently pass to the Leavitt path algebra as the actual algebra of interest associated to our quiver E. Concretely, it is given by a quotient algebra of $k\hat{E}$ modulo the Cuntz-Krieger relations that are compatible both with the *-structure and with the $\mathbb{Z}$-grading. In case of $k = \mathbb{C}$ and from the viewpoint of noncommutative geometry, the Leavitt path algebra serves as an algebraic core for the even more structured graph C^*-algebra. Its construction additionally involves the completion in a C^*-norm as an analytic aspect.

Definition 3.6 ([2, Def. 1.2.3])**.** Let E be a quiver. Let $X \subseteq \mathrm{Reg}(E)$ be a set of regular vertices and let $k\hat{E}$ be the path algebra of the extended quiver over the field k. Then the quotient algebra by the *Cuntz-Krieger relations*

$$e^* f = \delta_{e,f}\, r(e) \quad \forall\, e, f \in E^1, \tag{CK1}$$

$$\sum_{e \in s^{-1}(v)} e e^* = v \quad \forall\, v \in X \tag{CK2}$$

is called *relative Leavitt path algebra* $L_k^X(E)$. Put differently, it is the result of the following adjunctions for kE:

$$L_k^X(E) = kE[\,(e^*)_{e \in E^1} \mid (E^*),\, (CK1),\, (CK2)].$$

If all regular vertices are chosen, that is, $X = \mathrm{Reg}(E)$, then X is dropped in the notation and we just speak of the *Leavitt path algebra* $L_k(E)$ for E.
If no vertices are chosen, that is, $X = \emptyset$, we refer to $C(E) := L_k^{\emptyset}(E)$ as the *Cohn algebra* of E. In fact, in [2], relative Leavitt path algebras are called "relative Cohn algebras" throughout.

Proposition 3.7 ([2, Prop. 1.5.5])**.** *Let E be a quiver and let $X \subseteq \mathrm{Reg}(E)$ be a set of regular vertices. For any relative Leavitt path algebra $L_k^X(E)$ the family of compatible path* *and ghost path products*

$$\mathcal{B} := \{\mu\nu^* \mid \mu, \nu \in \mathrm{Path}(E),\, r(\mu) = r(\nu)\}$$

generates it as a k-vector space. If no regular vertices are chosen, that is, $X = \emptyset$, then $\mathcal{B}$ provides a basis for the Cohn algebra $C(E)$.

Proof. Recall from Definition 2.28 that $\mathrm{Path}(\hat{E})$ forms a basis for the extended path algebra $k\hat{E}$ and that any $L_k^X(E)$ in Definition 3.6 arises as a quotient algebra of it. The $(CK1)$ relation enables us to simplify any confrontation e^*f of a ghost edge on the left and an edge on the right. In fact, it can either be left out for $e = f$ or yields zero otherwise. Thus, any path consisting of edges and ghost edges can be successively turned into a reduced form in $\mathcal{B} \cup \{0\}$ modulo $(CK1)$. This argument shows the first part.

For the second part, we have to show linear independence if no further $(CK2)$ relation is imposed, that is, if $X = \emptyset$. Let A be the vector space with formal basis $\mathcal{B}$. It comes with a canonical linear map $A \to C(E)$ that is surjective by the first part. The strategy for the rest of the proof is to show that A already is an algebra itself in order to get that the linear map above is in fact an algebra embedding at the same time.

Concretely, we aim to show that the bilinear map $() \cdot ()\colon A \times A \to A$ induced by the reduced concatenations in $\mathcal{B}$ is associative:

$$(\mu_1\nu_1^*) \cdot (\mu_2\nu_2^*) = \begin{cases} \mu_1\mu_2'\nu_2^*, & \exists\, \mu_2' \in \mathrm{Path}(E)\colon \mu_2 = \nu_1\mu_2', \\ \mu_1(\nu_1')^*\nu_2^*, & \exists\, \nu_1' \in \mathrm{Path}(E)\colon \nu_1^* = (\nu_1')^*\mu_2^*, \\ 0, & \text{otherwise.} \end{cases}$$

This involves a bunch of case distinctions in the computation of the two expressions

$$\begin{aligned} x &= (\mu_1\nu_1^*) \cdot [(\mu_2\nu_2^*) \cdot (\mu_3\nu_3^*)], \\ y &= [(\mu_1\nu_1^*) \cdot (\mu_2\nu_2^*)] \cdot (\mu_3\nu_3^*). \end{aligned}$$

The difference is that for y the reduction first occurs at $\nu_1^*\mu_2$, while for x the reduction at $\nu_2^*\mu_3$ is carried out first. As long as at least one of the inner paths μ_2 or ν_2^* absorbs its outer counterpart, these two reductions clearly do not interfere and yield $x = y$. The only somewhat exciting comparison of x and y occurs if the whole middle word gets absorbed, that is, $\nu_1^* = (\nu_1')^*\mu_2^*$ and $\mu_3 = \nu_2\mu_3'$. In this case, however, both computations reduce

$$(\nu_1')^*\mu_2^* \cdot \mu_2\mu_3' = (\nu_1')^*\nu_2^* \cdot \nu_2\mu_3' = (\nu_1')^* \cdot \mu_3'$$

in their second step, which decides whether a path or a ghost path survives between $\mu_1(.)\nu_3^*$ in the end. This shows the claim. □

Corollary 3.8 ([2, p. 37])**.** *Any relative Leavitt path algebra $L_k^X(E)$ inherits both the $\mathbb{Z}$-grading and the *-structure from the path algebra $k\hat{E}$ of the extended graph introduced in Definition 3.4. For $m \in \mathbb{Z}$ the subspace of m-homogeneous elements takes the shape of*

$$L_k^X(E)_m = \operatorname{span}_k\{\mu\nu^* \in \mathcal{B} \colon |\mu| - |\nu| = m\}.$$

Proof. We can implement the Cuntz-Krieger relations iteratively. By Definition 3.6 the Cohn algebra $C(E)$ is the quotient of $k\hat{E}$ modulo the ideal

$$k\hat{E}\{e^*f - \delta_{e,f}r(e) \mid e, f \in E^1\}k\hat{E}$$

associated to the first Cuntz-Krieger relation $(CK1)$. Since this ideal is generated by 0-homogeneous and *-invariant elements, both the $\mathbb{Z}$-grading and the *-structure pass on to the quotient algebra $C(E)$. From here, the relative Leavitt path algebra $L_k^X(E)$ is obtained by implementing the second Cuntz-Krieger relation as well. Therefore, it is the quotient of $C(E)$ modulo the ideal

$$C(E)\{x - \sum_{e \in s^{-1}(x)} ee^* \mid x \in X\}C(E).$$

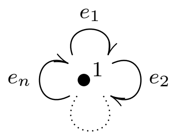
This ideal is also generated by 0-homogeneous and *-invariant elements. As a result, the structure passes down to $L_k^X(E)$ as well. Now, the claim follows from Proposition 3.7 since by Definition 3.4 we have for every $\mu\nu^* \in \mathcal{B}$ that

$$\deg(\mu\nu^*) = \deg(\mu) + \deg(\nu^*) = |\mu| - |\nu|.$$

The path length contributes with a positive sign, while the ghost path length gets subtracted. □

Example 3.9 ([2, p. 11]). Recall the quiver

$$R_n =$$



from Example 2.30.
Its path algebra is the free unital algebra in n generators. If we choose the only available vertex for $X = \{1\}$, then the Cuntz-Krieger relations take the following shape:

$$e_i^* e_j = \delta_{i,j}, \quad i, j \le n, \qquad \sum_{j \le n} e_j e_j^* = 1.$$

With an operator analytic eye on these relations, they describe that the generators are n isometries in algebraic terms, whose orthogonal source projections sum up to 1. In this context, however, the assumption that e_j is a bounded operator on a separable Hilbert space is not imposed. Therefore, these algebras merely serve as *algebraic Cuntz algebras*. For $n \ge 2$, they coincide with Leavitt's family of counterexamples for the invariant basis number property from the introduction. For $n = 1$, on the other hand, the ghost edge serves as a two-sided inverse and

$$L_k(R_1) \cong k[x, x^{-1}]$$

behaves quite differently. It yields the algebra of Laurent polynomials with coefficients in k. The graph theoretic difference is that the loop in R_1 has no exit, while R_n offers alternative loops for $n \ge 2$.
If we do not impose $(CK2)$ on the vertex in R_1, however, then we get the Cohn algebra $L_k^{\emptyset}(R_1)$, where $s := e_1$ is only supposed to have a left inverse adjoint s^*. In other words, this corresponds to a single isometry instead of a single unitary in the context of operator algebras. Hence, the resulting Cohn algebra

$$C(R_1) \cong k[s, s^* \mid s^* s = 1]$$

serves as an *algebraic Toeplitz algebra*.

Example 3.10 ([2, p. 11], [1, p. 14]). The following quivers all produce matrix algebras

$M_n(k)$ as their Leavitt path algebras:

$$A_n = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n},$$

$$\begin{tikzcd} & \bullet^{w_1} \arrow[d, "e_1"] & \\ B_n = \bullet^{w_{n-1}} \arrow[r, "e_{n-1}"] & \bullet^{v} & \arrow[l, dotted] \\ & \arrow[u, dotted] & \end{tikzcd} ,$$

$$\begin{tikzcd} D_n = \bullet^{v} \arrow[r, bend left, "e_1"] \arrow[r, dotted] \arrow[r, bend right, "e_{n-1}"'] & \bullet^{w} \end{tikzcd} .$$

At first glance, the quivers seem to behave rather differently, but at second sight, they all share a common path structure. Indeed, there are precisely $n-1$ paths that all arrive at the unique sink v_n, v or w in the end. They give rise to the respective isomorphisms to $M_n(k)$ with the paths filling the upper off-diagonals, their ghost paths filling the lower ones, and vertices as the diagonal matrix units.
Concretely, for A_n, this takes the shape of

$$\begin{aligned} L_k(A_n) &\to M_n(k) \\ v_j &\mapsto E_{j,j} \\ e_j &\mapsto E_{j,j+1} \\ e_j^* &\mapsto E_{j+1,j}. \end{aligned}$$

Note that in this picture the transition to $M_\infty(k) = \bigcup_{n\in\mathbb{N}} M_n(k)$ in Example 2.10 can be visualised by the underlying quivers. Either by extending the line quiver A_n infinitely to the right or by equipping the clock quiver B_n with infinitely many arms. In D_n, the analogous procedure would turn the vertex v in D_n into an infinite emitter, though. So the formal quiver D_∞ admits a different regular vertex structure than all of the quivers D_n. In fact, unlike as for the first two quiver sequences, this form is not well-suited for direct limit constructions since the fibre $s^{-1}(v)$ gets manipulated along the way.
See also [2, Example 1.6] for more details.

Example 3.11 ([2, Example 1.5.18], [1, p. 15]). The last example illustrates that the underlying quiver for a relative Leavitt path algebra is by no means unique. There is even a standard way to realise any relative Leavitt path algebra $L_k^X(E)$ as the honest Leavitt path algebra $L_k(E(X))$ for a bigger quiver $E(X)$. It is constructed by copying all unchosen regular vertices $\widetilde{Y} = \{\tilde{v} \mid v \in \mathrm{Reg}(E) \setminus X\}$ as well as all edges

$$\{\tilde{e}\colon s(e) \to \widetilde{r(e)} \mid e \in r^{-1}(\mathrm{Reg}(E) \setminus X)\}$$

leading to them. Thus, $E(X)$ extends the old quiver E in such a way that a new vertex $\tilde{v} \in \tilde{Y}$ becomes a sink, which receives copies of all edges that point towards the original vertex v. Applied to the algebraic Toeplitz algebra from Example 3.9 with $E = R_1$ and $X = \emptyset$, this procedure appends an extra vertex $w = \tilde{v}$ and an edge $f = \tilde{e}$:

$$E_T = \begin{tikzcd} \bullet^v \arrow[loop left, "e"] \arrow[r, "f"] & \bullet^w \end{tikzcd} .$$

In this case, we can verify by hand that the following assignments

$$\begin{aligned} \varphi\colon k[s, s^* \mid s^*s = 1] &\to L_k(E_T) \\ s &\mapsto e + f, \\ s^* &\mapsto e^* + f^* \end{aligned}$$

lead to a graded algebra isomorphism. It is well-defined since the grading is preserved and we have

$$(e^* + f^*)(e + f) = u + 0 + 0 + v = 1_{L_k(E_T)}.$$

Moreover, the elements

$$v' := ss^*, \quad w' := 1 - ss^*, \quad e' := s^2s^*, \quad f' := s - s^2s^*$$

provide a family in $k[s, s^* \mid s^*s = 1]$ that preserves the grading and satisfies the defining relations for $L_k(E_T)$. Indeed, v' and w' are clearly orthogonal idempotents with

$$v'e'v' = s(s^*s)s(s^*s)s^* = e', \quad v'f'w' = s(s^*s)(1 - ss^*) - s(s^*s)ss^*(1 - ss^*) = f' - 0.$$

Furthermore, the Cuntz-Krieger relations are satisfied since

$$\begin{aligned} (e')^*e' &= ss^{*2}s^2s^* = v', & (f')^*f' &= (s^* - ss^{*2})(s - s^2s^*) = 1 - ss^* - ss^* + ss^* = w', \\ (e')^*f' &= ss^{*2}s - v' = 0, & e'(e')^* + f'(f')^* &= s^2s^{*2} + (s - s^2s^*)(s^* - ss^{*2}) = ss^* = v'. \end{aligned}$$

Thus, the $()'$-assignment extends to a graded algebra homomorphism ψ in the inverse direction and we have that $\psi \circ \varphi$ is the identity by construction of $e' + f' = s$. Likewise, also $\varphi \circ \psi$ is the identity because of the two main computations

$$\varphi\psi(v) = (e+f)(e^* + f^*) = ee^* + ff^* = v, \quad \varphi\psi(e) = (e+f)^2(e^* + f^*) = (e+f)v = e.$$

This finally shows the claimed isomorphism. Interestingly, with the previous examples in mind, the quiver E_T already suggests that the corresponding Leavitt path algebra has both a matrix algebra and a Laurent part.

3.2 Cuntz-Krieger relations as invertibility conditions

Having established both a few basic properties and examples of relative Leavitt path algebras, we can turn to Cohn localisations again.

Remark 3.12. Note that our route to relative Leavitt path algebras starts with Lemma 3.3 at the beginning of the preceding section. It motivates to introduce the notion of ghost edges in Definition 3.4 at all. In the context of Lemma 3.3, however, only the ghost edges for $s^{-1}(X)$ show up. They arise as the adjoined generators for a Cohn localisation of the path algebra kE at regular path prolongation maps. Furthermore, the matrix invertibility conditions turn out to be equivalent to a weak form of the Cuntz-Krieger relations limited to $s^{-1}(X)^*$.
So the Cohn localised path algebra $\mathrm{Cohn}^X(E)$ already has a remarkable similarity to $L_k^X(E)$ and both constructions even coincide once we have $E^1 = s^{-1}(X)$. With the notation from Definition 3.1 the decomposition

$$E^1 = s^{-1}(\mathrm{Reg}(E) \cup \mathrm{Inf}(E))$$

shows that this happens precisely for the Leavitt path algebra of a row-finite quiver, that is, $X = \mathrm{Reg}(E)$ and $\mathrm{Inf}(E) = \emptyset$.
However, apart from this special case, the relative Leavitt path algebra additionally contains the remaining ghost edges $\{e^* \mid e \in s^{-1}(E^0 \setminus X)\}$, for which the weak relations $(w.E^*)$ and $(w.CK1)$ get extended to (E^*) and $(CK1)$.

Inspired by this, our next goal is to realise all relative Leavitt path algebras as Cohn localisations to benefit from the Cohn localisation theory we have elaborated so far.
As a first attempt, we should consider the remaining joint multiplication maps $(\varphi^v)_{v \notin X}$

from (11). Actually, sinks can be discarded immediately since they do not have any edges that start from it. What remains is the index set

$$E^0 \setminus (X \cup \mathrm{Sink}(E)) = (\mathrm{Reg}(E) \setminus X) \cup \mathrm{Inf}(E)$$

of unchosen regular vertices and infinite emitters.

Lemma 3.13. *Let* $\mathrm{Cohn}^X(E)$ *be as in Definition 3.2. Let* L *be the algebra that is obtained from it by adjoining generators* $\{e^* \mid e \in s^{-1}(\mathrm{Reg}(E) \setminus X) \cup s^{-1}(\mathrm{Inf}(E))\}$ *that are designed to form "right inverses" of the induced joint multiplication homomorphisms*

$$\Phi^v \colon Lv \to \prod_{e \in s^{-1}(v)} Lr(e)$$

for $v \in (\mathrm{Reg}(E) \setminus X) \cup \mathrm{Inf}(E)$ *in the sense that*

$$\Psi^v := \sum_{e \in s^{-1}(v)} () \cdot e^* \colon \bigoplus_{e \in s^{-1}(v)} Lr(e) \to Lv$$

satisfies $\Phi^v \circ \Psi^v = \mathrm{Id}_{\bigoplus_{e \in s^{-1}(v)} Lr(e)}$*. Then* L *is the relative Leavitt path algebra* $L_k^X(E)$*.*

Proof. The final part of the proof of Lemma 3.3 serves as a guideline. Indeed, it already covers the discussion for the by now unconsidered regular vertices in $\mathrm{Reg}(E) \setminus X$. For Φ^v with $v \in \mathrm{Reg}(E) \setminus X$, the proof tells us that the postulated right inverse over the constructed algebra is formed by ghost edges for $s^{-1}(v)$ subject to the $(CK1)$ relation. In fact, this argument even works out for infinite emitters $v \in \mathrm{Inf}(E)$ as Ψ^v is only supposed to live on the direct sum. Concretely, since $\bigoplus$ serves as the coproduct on the category of modules, any module homomorphism

$$\bigoplus_{e \in s^{-1}(v)} Lr(e) \to Lv$$

is given by a sum of module homomorphisms $Lr(e) \to Lv$, even for an infinite index set. For any sequence of elements, the involved sum is well-defined due to its finite support. The component left module homomorphisms are pinned down by the image of the right unit $r(e)$. In other words, the postulated homomorphism is necessarily of the shape

$$\Psi^v = \sum_{e \in s^{-1}(v)} () \cdot e^* \colon \bigoplus_{e \in s^{-1}(v)} Lr(e) \to Lv$$

for some $e^* \in r(e)Ls(e)$. Likewise, any endomorphism on this direct sum is pinned down by the images of $(\delta_{f,e}r(e))_f$ for $e \in s^{-1}(v)$. This yields the equivalences

$$\begin{aligned}
\Phi^v \circ \Psi^v = \mathrm{Id}_{\bigoplus_{f\in s^{-1}(v)} Lr(f)} &\iff \Phi^v \circ \Psi^v(\delta_{f,e}r(e))_f = (\delta_{f,e}r(e))_f \quad \forall\, e \in s^{-1}(v) \\
&\iff (r(e)e^*f)_f = (\delta_{f,e}r(e))_f \quad \forall\, e \in s^{-1}(v) \\
&\iff e^*f = \delta_{f,e}r(e) \quad \forall\, e,f \in s^{-1}(v) \\
&\iff e^*f = \delta_{f,e}r(e) \quad \forall\, e \in s^{-1}(v),\ f \in E^1.
\end{aligned}$$

So the additional generators in L form a right inverse if and only if they satisfy both (E^*) and $(CK1)$. This shows the claim. □

This right invertibility rephrasement in Lemma 3.13 for the part of the Cuntz-Krieger relations that we have not expressed in terms of a Cohn localisation so far is not yet completely satisfactory. Actually, we would like to realise any relative Leavitt path algebra $L_k^X(E)$ as a Cohn localisation of a simpler algebra B by modelling all ghost edges as inverses for suitable edge multiplication maps $() \cdot e$. Unfortunately, the candidate maps of joint fibrewise multiplication we have used so far in Lemma 3.13 do not work for edges $e \notin s^{-1}(X)$. The problem is that they are not even left invertible over the relative Leavitt path algebra because of the lack of a $(CK2)$ relation involving edges $e \notin s^{-1}(X)$.
This suggests to refrain from dealing with joint multiplication and to look for separated multiplication maps $() \cdot e$ for $e \notin s^{-1}(X)$ instead that already admit a built-in left invertibility over $L_k^X(E)$. In fact, multiplication with e^* is left inverse to multiplication with e if and only if the multiplication $() \cdot ee^*$ acts trivially on the domain. Thus, our planned Cohn localisation map relies on an unlocalised algebra that already admits corners for formal elements that take the shape ee^*, once the ghost edge e^* is available. As some sort of backwards engineering, it therefore seems appropriate to learn more about elements of the shape ee^* and their relations in $L_k^X(E)$ in order to implement them correctly in the unlocalised algebra.

Lemma 3.14. *Let $L_k^X(E)$ be a relative Leavitt path algebra and write $p_e := ee^*$ for the so called source projection of $e \in E^1$. Given two edges $e, f \in E^1$, they have the following properties:*

$$p_e p_f = \delta_{e,f} p_e, \quad p_e \leq s(e), \quad p_e e = e, \quad e^* p_e = e^*.$$

Both Cuntz-Krieger relations can be interpreted in this context: $(CK2)$ says that the *source projections* $(p_e)_{e\in s^{-1}(x)}$ for a regular vertex $x \in X$ sum up to their common source

x, while $(CK1)$ directly identifies any *range projection* e^*e with its range vertex $r(e)$ and ensures the listed properties above.

Proof. Let $e, f \in E^1$ be two edges. Then $(CK1)$, (E) and (E^*) imply all listed statements:

- $p_e p_f = e(e^*f)f^* = e(\delta_{e,f}r(e))f^* = \delta_{e,f}er(e)e^* = \delta_{e,f}p_e$,
- $s(e)p_e = s(e)ee^* = ee^* = ee^*s(e) = p_e s(e)$,
- $p_e e = e(e^*e) = er(e) = e$,
- $e^*p_e = (e^*e)e^* = r(e)e^* = e^*$. $\square$

Intuitively, source projections for edges in a common fibre $e \in s^{-1}(v)$ consistently refine both the initial vertex structure of kE and its invariance relation (E) for edges. More concretely, source projections come as orthogonal idempotents $(p_e)_{e\in s^{-1}(v)}$ with $p_e \leq v$ and as such they highlight direct summands

$$\bigoplus_{e\in s^{-1}(v)} L_k^X(E)p_e \subseteq L_k^X(E)v$$

of its vertex corner. Source projections provide an environment to capture a stronger invariance condition $p_e er(e) = e$ compared to (E) and therefore allow to restrict the vertex corner in the domain of

$$\big(() \cdot e\colon L_k^X(E)v \to L_k^X(E)r(e)\big)_{e\in s^{-1}(v)}$$

to the better behaved multiplication map

$$\big(\Phi^e := () \cdot e\colon L_k^X(E)p_e \to L_k^X(E)r(e)\big)_{e\in s^{-1}(v)}.$$

In fact, the multiplication map $() \cdot e^*$ provides a two-sided inverse for every Φ^e by design of $p_e = ee^*$ and $e^*e = r(e)$ in $L_k^X(E)$. This motivates to work around the one-sided localisation in Lemma 3.13 by adjoining formal auxiliary source projections p_e for the critical edges $e \in s^{-1}(E^0 \setminus X)$ to our path algebra kE first. This provides a generalised path algebra B, which allows for Cohn localisations both at the usual regular joint multiplication maps and at the restricted edge multiplication maps starting from Bp_e.

Definition 3.15. Let E be a quiver and let $X \subseteq \operatorname{Reg}(E)$ be a selection of regular vertices. Extend the path algebra kE by adjoining the set of generators

$$S := \{p_e \mid e \in s^{-1}(E^0 \setminus X)\}$$

subject to the following relations for $e, f \in s^{-1}(E^0 \setminus X)$:

$$(SP) \quad p_e p_f = \delta_{e,f} p_e, \quad p_e \leq s(e), \quad p_e e = e.$$

This defines the so called *generalised path algebra*

$$B := kE[(p_e)_{e \in s^{-1}(E^0 \setminus X)} \mid (SP)].$$

All source projections $p_e \in S$ are invariant under their respective source vertex $s(e)$, so the vertices E^0 that are known to span the local units in kE also do the job in B. There are induced collections of left B-module homomorphisms that all suit for Cohn localisation maps: On the one hand the joint multiplication maps

$$\left(\varphi^v \colon Bv \to \bigoplus_{e \in s^{-1}(v)} Br(e)\right)_{v \in X}$$

for the chosen regular vertices in X and on the other hand the restricted edge multiplication maps

$$(\varphi^e \colon Bp_e \to Br(e))_{e \in s^{-1}(E^0 \setminus X)}$$

for the remaining edges.

Theorem 3.16. *Let B be a generalised path algebra as in Definition 3.15. Then its Cohn localisation at the localisation maps $(\varphi^v)_{v \in X}$ and $(\varphi^e)_{e \in s^{-1}(E^0 \setminus X)}$ is the corresponding relative Leavitt path algebra:*

$$\operatorname{Cohn}(B) = L_k^X(E).$$

Both B and $\operatorname{Cohn}(B)$ *have enough idempotents and are unital if and only if the underlying quiver E has finitely many vertices.*

Proof. Both algebras are successively built from the path algebra kE by adjoining additional generators subject to given relations. For the construction of B, we first adjoin source projections p_e for $e \in s^{-1}(E^0 \setminus X)$ subject to the relations (SP). Now, for every

$v \in X$ and $e \in s^{-1}(E^0 \setminus X)$ the localisation leads to invertible maps Φ^v and Φ^e over the Cohn localisation. Explicitly, we adjoin Cohn ghost edges $(f_B^*)_{f \in s^{-1}(v)}$ and e_B^* that are modelled to form inverses for Φ^v and Φ^e, respectively. On the one hand, Lemma 3.13 already shows that the invertibility conditions for the ghost edges f_B^* for $f \in s^{-1}(X)$ are equivalent to $(w.E^*)$ and both Cuntz-Krieger relations $(w.CK1)$ and $(CK2)$.
On the other hand, we also know that $(\Phi^e)^{-1} \colon \mathrm{Cohn}(B)r(e) \to \mathrm{Cohn}(B)p_e$ is a multiplication map by some $e_B^* \in r(e)\mathrm{Cohn}(B)p_e$ while bijectivity leads to the equivalent relations $ee_B^* = p_e$ and $e_B^* e = r(e)$. Note, however, that it is already sufficient to demand $e_B^* \in r(e)\mathrm{Cohn}(B)s(e)$ given bijectivity due to

$$e_B^* p_e = (e_B^* e)e_B^* = r(e)e_B^* = e_B^*.$$

Having established these defining relations, the claim reads as

$$\begin{aligned}
\mathrm{Cohn}(B) &= B[(e_B^*)_{e \in E^1} \mid (\Phi^e)^{-1} = () \cdot e_B^*, (\Phi^v)^{-1} = \sum_{e \in s^{-1}(v)} () \cdot e_B^*] \\
&= kE[(e_B^*)_{e \in E^1} \mid p_e = ee_B^*, (SP), e_B^* e = r(e)\, \forall e \notin s^{-1}(X), (E^*), (w.CK1), (CK2)] \\
&\overset{!}{=} kE[(e^*)_{e \in E^1} \mid (E^*), (CK1), (CK2)] \\
&= L_k^X(E)
\end{aligned}$$

and it is left to show equality of the relations that are imposed on the Cohn ghost edges e_B^* in the second and on the Leavitt path algebra ghost edges e^* in the third line. This equivalence is essentially due to our backwards engineering of the source projection relations (SP): By Lemma 3.14 the source projections $ee^* \in L_k^X(E)$ clearly satisfy (SP). Thus, the relations in the third imply the relations in the second line. Conversely, the Cohn ghost edges for $e \in s^{-1}(E^0 \setminus X)$ also satisfy the missing $(CK1)$ relation by orthogonality in (SP):

$$e_B^* f = r(e)e_B^* f r(f) = (e_B^* e)e_B^* f(f_B^* f) = e_B^* p_e p_f f = e_B^* f \delta_{e,f} = e_B^* e \delta_{e,f} = r(e)\delta_{e,f}.$$

This shows equality. Finally, all adjoined ghost edges in the Cohn localisation also have a source and a range vertex. In particular, we still have that the vertices span a set of local units as in Definition 2.8. Therefore, B and $L_k^X(E)$ are unital if and only if the set of vertices is finite and in this case $\sum_{v \in E^0} v$ serves as a unit element. □

This discussion finally establishes all relative Leavitt path algebras as Cohn localisations and provides the opportunity to import results from the unlocalised algebra B that was constructed for this purpose. Indeed, dealing with B is not much worse than dealing with the path algebra since its structure is mainly defined by families of orthogonal idempotents, too.

Proposition 3.17. *Let B be a generalised path algebra as in Definition 3.15. Then B is quasi-free.*

Proof. Let $\pi\colon \hat{B} \to B$ describe a square-zero extension of B. By Theorem 2.50 the path algebra kE is quasi-free and this allows to lift the natural inclusion $kE \to B$ along π using Theorem 2.36. However, this merely provides an appropriate lift for the vertices and does not treat source projections yet. Moreover, the lifts of the edges possibly need to be adapted because of the corner membership $e \in p_e B r(e)$ in case of $s(e) \notin X$. The essential part of the proof in Theorem 2.50 was to lift the vertices as a countable family of orthogonal idempotents. This iterative lifting procedure allows to weave in the countable family of source projections S as well. For this, let us describe one layer of iteration explicitly.
At the point where we lift a new vertex $v \in E^0$ to an idempotent $\hat{v}$ that is orthogonal to all previously lifted vertices, we also look whether there are source projections associated to its fibre. In this case, that is, if $v \in E^0 \setminus (\mathrm{Sink}(E) \cup X)$, we can start a new orthogonal lifting process for the first of the source projections $(p_e)_{e \in s^{-1}(v)}$ in its countable fibre. In fact, we can even take care of the correct corner membership simultaneously and find an appropriate lift $\hat{p}_e \leq \hat{v}$ by Corollary 2.49. Finally, we complete this layer by continuing with the next respective iteration in all of the finitely many processes associated to vertices of this kind that have already been lifted earlier.

$$\begin{tikzcd}
v \notin X \arrow[r, dash, "\perp"] \arrow[d, dash, "\leq"] & w \notin X \arrow[r, dash, "\perp"] \arrow[d, dash, "\leq"] & x \in X \arrow[r, dash, "\perp"] \arrow[d, dash, "\leq"] & \dots \\
(p_{e_1} \perp p_{e_2} \perp \dots)_{s(e_j)=v} & (p_{f_1} \perp p_{f_2} \perp \dots)_{s(f_j)=w} & \text{none} & \dots
\end{tikzcd}$$

In this fashion we exhaust the whole countable idempotent structure of vertices and source projections and manage to lift it to idempotents $\{\hat{x} \mid x \in E^0 \cup S\} \subseteq \hat{B}$ such that orthogonality and the relations $\hat{p}_e \leq \hat{s}(e)$ are respected:

$$\hat{v}\hat{w} = \delta_{v,w}\hat{v} \quad \forall\, v, w \in E^0,$$
$$\hat{p}_e\hat{p}_f = \delta_{e,f}\hat{p}_e \quad \forall\, e, f \in s^{-1}(E^0 \setminus X).$$

Having done that, both the edges $e \in p_e B r(e)$ starting outside of X and those $e \in s(e) B r(e)$

starting inside X can now be lifted properly as in the proof of Theorem 2.50 to some $\hat{e} \in \hat{p}_e \hat{B} \hat{r}(e)$ or $\hat{e} \in \hat{s}(e) \hat{B} \hat{r}(e)$, respectively. In the end, the assignment $\hat{(.)}$ lifts all generators in B to elements in $\hat{B}$ with the same relations (SP) and thus extends to a split algebra homomorphism $B \to \hat{B}$ by the universal property of B. $\square$

Corollary 3.18. *All relative Leavitt path algebras are quasi-free.*

Proof. By Theorem 3.16 they can be written as Cohn localisations of generalised path algebras which are quasi-free by Proposition 3.17. Now apply Theorem 2.55. $\square$

4 Resolutions for Leavitt path algebras

We have already met derivations in the analysis of sections for the crossed product algebra in Definition 2.44 and Theorem 2.45. To have yet another alternative description of derivations for an algebra A, we introduce the bimodule of noncommutative forms $\Omega^1(A)$, too. It is the kernel of the multiplication map starting at $A \otimes A$ and captures derivations in terms of bimodule homomorphisms.

At this point, in turns out that the results for a Cohn localised algebra can already be obtained from the unlocalised algebra. In context of a relative Leavitt path algebra L, we can therefore phrase $\Omega^1(L)$ in terms of $\Omega^1(B)$ with B as its underlying generalised path algebra. In the end, we want to benefit from this in the computation of an explicit projective bimodule resolution of L. This approach is slightly different from the one in [9], which computes the kernel of the multiplication map starting at $L \otimes_{k[E^0]} L$ instead of $L \otimes L$, which is sufficient as long as the underlying quiver is row-finite.

4.1 Bimodules of noncommutative forms

Lemma 4.1. *Any derivation D from a unital algebra into a unital bimodule annihilates the unit element. In particular, if M is any bimodule over an algebra A that is turned into a unital A^+-bimodule, then there is a canonical bijection*

$$\begin{aligned} \mathrm{Der}(A^+, M) &\cong \mathrm{Der}(A, M) \\ D &\mapsto D|_A \\ (D', 0) &\leftarrow\!\shortmid D'. \end{aligned}$$

Proof. We have

$$D(1) = D(1 \cdot 1) = 1 \cdot D(1) + D(1) \cdot 1 = 2D(1) \implies D(1) = 0.$$

In particular, if $D \in \mathrm{Der}(A^+, M)$ is an A^+-derivation into a unital A^+-bimodule M, then its k-component is forced to be zero, while the A-component map is linear and inherits the Leibniz rule by design. This shows the claim. ☐

Definition 4.2 ([24, Def. 15.1])**.** Let A be an algebra. The kernel

$$\overline{\Omega^1}(A) := \ker(\mathrm{mult}\colon A \otimes A \to A)$$

of the multiplication map for A is called the *reduced bimodule of noncommutative forms*. If A is self-induced, then mult is surjective and we obtain a short exact sequence of A-bimodules by design:

$$0 \to \overline{\Omega^1}(A) \to A \otimes A \to A \to 0. \tag{12}$$

If A even has a unit element 1_A, then the map

$$\begin{aligned} d_A := \mathrm{ad}_{1_A \otimes 1_A} \colon A &\to \overline{\Omega^1}(A) \\ a &\mapsto 1_A \otimes a - a \otimes 1_A \end{aligned}$$

is called *reduced universal derivation*. If we apply these definitions to the unitalisation A^+, then the word "reduced" is omitted. That is, the kernel $\Omega^1(A) := \overline{\Omega^1}(A^+)$ is called the *bimodule of noncommutative forms*, while the map

$$\begin{aligned} d := d_{A^+} \colon A^+ &\to \Omega^1(A) \\ a &\mapsto 1 \otimes a - a \otimes 1 \end{aligned}$$

is called *universal derivation*.

Proposition 4.3 ([24, Prop. 15.2])**.** *Let $\tilde{A}$ be a unital algebra. The map $d_{\tilde{A}}$ in Definition 4.2 is indeed a well-defined derivation. It allows to characterise the reduced bimodule of noncommutative forms as*

$$\overline{\Omega^1}(\tilde{A}) = \mathrm{span}_k\{ad_{\tilde{A}}(b) \mid a \in \tilde{A},\, b \in \tilde{A}/k\} \cong_{d_{\tilde{A}} \mapsto \otimes} \tilde{A} \otimes \tilde{A}/k$$

with the adapted right module structure $(a \otimes b) \cdot c := a \otimes (bc) - ab \otimes c$ for $c \in \tilde{A}$ on $\tilde{A} \otimes \tilde{A}/k$. Furthermore, $d_{\tilde{A}}$ is the universal $\tilde{A}$-derivation into a unital $\tilde{A}$-module in the sense that any other derivation $D \in \mathrm{Der}(\tilde{A}, \tilde{M})$ into a unital bimodule $\tilde{M}$ factors through it by a unique $\tilde{A}$-bimodule homomorphism. Phrased diagrammatically:

$$\begin{tikzcd} \tilde{A} \arrow[r, "d_{\tilde{A}}"] \arrow[dr, "D"'] & \overline{\Omega^1}(\tilde{A}) \arrow[d, dotted, "\exists!\,\mathrm{hom}"] \\ & \tilde{M}. \end{tikzcd}$$

In particular, if $\tilde{A} = A^+$ is the unitalisation of some algebra A, then these results take the following shape:

We have $d \in \mathrm{Der}(A^+, \Omega^1(A)) \cong \mathrm{Der}(A, \Omega^1(A))$ *and the bimodule of noncommutative forms is characterised as*

$$\Omega^1(A) = \mathrm{span}_k\{ad(b) \mid a \in A^+, b \in A\} \cong_{d \mapsto \otimes} A^+ \otimes A.$$

Furthermore, d *is the universal* A*-derivation in the sense that any other derivation* $D \in \mathrm{Der}(A, M)$ *into a bimodule* M *factors through it by a unique bimodule homomorphism. Phrased diagrammatically:*

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "D"'] & \Omega^1(A) \arrow[d, dotted, "\exists! \text{ hom}"] \\ & M. \end{tikzcd}$$

Proof. The assignment in Definition 4.2 is well-defined because of $1_{\tilde{A}} \cdot a = a \cdot 1_{\tilde{A}}$ for all $a \in \tilde{A}$. It also inherits linearity from the tensor product and satisfies the Leibniz rule for $a, b \in \tilde{A}$:

$$d_{\tilde{A}}(ab) = 1_{\tilde{A}} \otimes ab - ab \otimes 1_{\tilde{A}} = (1_{\tilde{A}} \otimes a - a \otimes 1_{\tilde{A}})b + a(1_{\tilde{A}} \otimes b - b \otimes 1_{\tilde{A}}) = d_{\tilde{A}}(a)b + ad_{\tilde{A}}(b).$$

Now, by Lemma 4.1 it annihilates $k \cdot 1_{\tilde{A}}$ and therefore factors through $\tilde{A}/k$. Since mult is an $\tilde{A}$-bimodule homomorphism, we therefore get

$$\overline{\Omega^1}(\tilde{A}) \supseteq \mathrm{span}_k\{ad_{\tilde{A}}(b) \mid a \in \tilde{A}, b \in \tilde{A}/k\}.$$

Conversely, any k-linear combination $\sum_j \lambda_j a_j \otimes b_j \in \overline{\Omega^1}(\tilde{A})$ satisfies $\sum_j \lambda_j a_j b_j = 0$ and can be rewritten in the desired shape:

$$\begin{aligned} \sum_j \lambda_j a_j \otimes b_j &= \sum_j \lambda_j a_j \otimes b_j - \sum_j \lambda_j a_j b_j \otimes 1_{\tilde{A}} \\ &= \sum_j \lambda_j a_j (1_{\tilde{A}} \otimes b_j - b_j \otimes 1_{\tilde{A}}) \\ &= \sum_j \lambda_j a_j d_{\tilde{A}}(b_j). \end{aligned}$$

This establishes an isomorphism $\overline{\Omega^1}(\tilde{A}) \cong \tilde{A} \otimes \tilde{A}/k$ as left modules by exchanging $\otimes$ for $d_{\tilde{A}}$. To use it as a bimodule isomorphism, however, the right module structure has to reflect the calculation $(ad_{\tilde{A}}(b)) \cdot c = ad_{\tilde{A}}(bc) - abd_{\tilde{A}}(c)$ for $a, c \in \tilde{A}$ and $b \in \tilde{A}/k$ based on the Leibniz rule in $\overline{\Omega^1}(\tilde{A})$, as claimed.

This idea of replacing the tensor product operation also works for any other derivation $D\colon \tilde{A} \to \tilde{M}$ into a unital bimodule $\tilde{M}$ and gives a left module homomorphism

$$\begin{aligned} F\colon \tilde{A} \otimes \tilde{A} &\to \tilde{M} \\ a \otimes b &\mapsto aD(b). \end{aligned}$$

In fact, F fails to be a right module homomorphism on $\tilde{A} \otimes \tilde{A}$ because

$$F(a \otimes bc) = aD(b)c + abD(c) = F(a \otimes b)c + abD(c)$$

for $c \in \tilde{A}$. But since the extra summand vanishes for elements $a \otimes b = ad_{\tilde{A}}(b) \in \overline{\Omega^1}(\tilde{A})$, at least on this submodule F restricts to a bimodule homomorphism $f := F|_{\overline{\Omega^1}(\tilde{A})}$, which satisfies

$$f \circ d_{\tilde{A}}(a) = F(1_{\tilde{A}} \otimes a) - F(a \otimes 1_{\tilde{A}}) = 1_{\tilde{A}} \cdot D(a) - aD(1_{\tilde{A}}) = D(a)$$

for all $a \in \tilde{A}$ by construction. At this point, we need that $\tilde{M}$ is unital both to use that $1_{\tilde{A}}$ acts trivially and to apply Lemma 4.1. We even have that f is unique since any other bimodule homomorphism $f'\colon \overline{\Omega^1}(\tilde{A}) \to \tilde{M}$ with $f' \circ d_{\tilde{A}} = D$ also needs to satisfy

$$f'(ad_{\tilde{A}}(b)) = af'(d_{\tilde{A}}(b)) = aD(b) = f(ad_{\tilde{A}}(b))$$

for $a \in \tilde{A}$, $b \in \tilde{A}/k$. This proves the universal property.
Applied to the special case of $(\tilde{A}, 1_{\tilde{A}}) = (A^+, 1)$ for some algebra A, these results take a simpler shape. First of all, we have $d \in \mathrm{Der}(A^+, \Omega^1(A))$ and Lemma 4.1 justifies to identify d with its restriction to A, which is an A-derivation. Secondly, $A^+/k = A$ immediately gives the result on the bimodule of noncommutative forms. Finally, the observation that unital A^+-modules are nothing but A-modules M, which are extended by the identity action for 1, establishes the desired reformulation of the universal property. □

Corollary 4.4. *For any algebra A and A-bimodule M, we have natural bijections*

$$\begin{aligned} \mathrm{Hom}_{A,A}(\Omega^1(A), M) &\cong \mathrm{Der}(A, M) \cong \mathrm{Split}(M \to M \rtimes A \to A) \\ f &\mapsto D = f \circ d \mapsto (D, \mathrm{Id}_A). \end{aligned}$$

that are established in Theorem 2.43 and Proposition 4.3.

Remark 4.5. Note that the universal derivation $d = \mathrm{ad}_{1\otimes 1}$ for A^+ on the one hand and its reduced pendant $d_A = \mathrm{ad}_{1_A\otimes 1_A}$ in case of a unital A on the other hand yield two conceptually different derivations. In fact, for $a \in A \setminus \{0\}$ we have:

$$d_A(a) = a \otimes 1_A - 1_A \otimes a = 1_A a \otimes 1 \cdot 1_A - 1_A \cdot 1 \otimes a 1_A = 1_A d(a) 1_A \neq d(a).$$

Also compare with Remark 2.13. We keep working with the universal derivation d since it is available for any algebra and also suits for a neat description of both $\Omega^1(A)$ and $\overline{\Omega^1}(A)$, at least if A is self-induced, as the next lemma shows.

Lemma 4.6. *Let A be a self-induced algebra. Both the bimodule of noncommutative forms and the reduced one can be described in terms of the universal derivation d. Their difference is that the bimodule of noncommutative forms is the A^+-span of the image of d, while the reduced one is its A-span:*

$$\Omega^1(A) = A^+ d(A) A^+ \subseteq A^+ \otimes A^+,$$
$$\overline{\Omega^1}(A) = A d(A) A \subseteq A \otimes A.$$

If A admits set of generators $G \subseteq A$, then the image of the universal derivation is spanned by its values on generators $d(x)$ for $x \in G$, that is,

$$d(A) = A^+ d(G) A^+.$$

Therefore, it suffices to consider the respective spans of $d(G)$ in this case.

Proof. The first equation $\Omega^1(A) = A^+ d(A) A^+$ immediately follows from Proposition 4.3 and from the fact that mult is a bimodule homomorphism. Now, for the second equation we use that A is an ideal in A^+, which implies $A(A^+ \otimes A^+)A = A \otimes A$. In particular, $Ad(A)A = A\Omega^1(A)A \subseteq \overline{\Omega^1}(A)$. The inverse containment is clear from $\overline{\Omega^1}(A) \subseteq \Omega^1(A)$ since A is self-induced and both modules are therefore smooth.
Finally, if A is generated by $G \subseteq A$, then $d(G) \subseteq d(A)$ directly implies $A^+ d(G) A^+ \subseteq d(A)$. Thus, it is left to show that the image of d is spanned by its values on generators. Since d is linear, we may concentrate on words of generators $d(x_1 \cdots x_n)$ with $x_j \in G$ for $j \leq n$. Now, the Leibniz rule breaks it down to an A^+-linear combination of $d(x_j)$, as claimed:

$$d(x_1 \cdots x_n) = \sum_{j \leq n} x_1 \cdots d(x_j) \cdots x_n. \qquad \square$$

4.2 Projective bimodule resolutions

By definition, $\Omega^1(A)$ arises from the bilinear multiplication in A^+, which induces the multiplication map

$$\mathrm{mult}\colon A^+ \otimes A^+ \to A^+$$

by the universal property of the tensor product. In this context, the bimodule of noncommutative forms extends the multiplication map to an exact sequence of bimodules

$$0 \to \Omega^1(A) \to A^+ \otimes A^+ \to A^+ \to 0, \tag{13}$$

which captures multiplicative information on A. Here, the bimodule

$$A^+ \otimes A^+ = A^+(1 \otimes 1)A^+$$

is in some sense better behaved than A itself since it has the special property to be projective. Indeed, any projective A-bimodule P is of particular interest as its associated Hom-functor $\mathrm{Hom}_{A,A}(P, -)$ is not only left but also right exact by design of its defining property. It is therefore desirable to approximate an algebra as a bimodule over itself by a similar exact sequence of projective bimodules. This idea is made precise with the notion of a projective resolution.

Definition 4.7 ([24, Def. 17.3])**.** Let A be an algebra. Let $P_\bullet = (P_n, d^P)_{n\in\mathbb{N}_0}$ be a chain complex of projective A-bimodules that vanishes for negative degrees, and let $\epsilon\colon P_0 \to A$ be a bimodule homomorphism such that the augmented chain complex

$$\cdots \longrightarrow P_2 \xrightarrow{d_2^P} P_1 \xrightarrow{d_1^P} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

is exact. Then $P_\bullet$ is called a *projective bimodule resolution* of A, while ϵ is called the *augmentation map*.

Unfortunately, $\Omega^1(A)$ is not projective in general. So the exact sequence above in (13) needs to be adapted on the left. Indeed, there is a standard procedure to obtain a projective bimodule resolution for a unital algebra. If the algebra of interest is not unital, then it can be applied to its unitalisation.

Definition 4.8 ([20, p. 12])**.** Let A be a unital algebra. Consider the free A-bimodules

$$\mathrm{Bar}_n(A) := \begin{cases} A \otimes A^{\otimes n} \otimes A, & n \in \mathbb{N}, \\ A \otimes A, & n = 0. \end{cases}$$

For any $n \in \mathbb{N}$ and $0 \leq i \leq n$ we introduce the *face maps* $d_i \colon \mathrm{Bar}_n(A) \to \mathrm{Bar}_{n-1}(A)$ as the linear maps induced by

$$d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) := a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

Their alternating sum

$$b' = b'_n := \sum_{j=0}^{n} (-1)^j d_j \colon \mathrm{Bar}_n(A) \to \mathrm{Bar}_{n-1}(A)$$

is a boundary map and defines a chain complex $\mathrm{Bar}_\bullet := (\mathrm{Bar}_n, b')_{n \in \mathbb{N}_0}$ of A-bimodules. It provides a projective bimodule resolution of A with the multiplication $\mathrm{mult} \colon A \otimes A \to A$ as augmentation map and is called the *bar resolution* of A.

Proof. First of all, every face map and hence also b' are bimodule homomorphisms since the outside factors a_0 and a_{n+1} stay in order. Note that the defining formula for b' also makes sense for $n = 0$ and yields the multiplication map

$$b'_0 := d_0 = \mathrm{mult} \colon \mathrm{Bar}_0(A) \to \mathrm{Bar}_{-1}(A) := A.$$

In order to prove that b' is a boundary map, we take a step back and go more into detail. In $\mathrm{Bar}_n(A)$, we can enumerate the tensor factors from zero to $n+1$. For a pure tensor, we can further think of the $\otimes$ symbol after the i-th entry as the i-th "separator". This allows us to interpret the face map d_i as removing the i-th separator. Now, consider the face maps for two indices $0 \leq i < j \leq n$. If we want to remove both the i-th and the j-th of the current separators and start with d_j, then the numbering until $i < j$ is unaffected. If we start with d_i, however, then the former j-th separator becomes the $(j-1)$-th separator afterwards. Hence we have the identity

$$d_i d_j = d_{j-1} d_i. \tag{14}$$

It can be used to find a cancellation theme in

$$\begin{aligned}
b'_{n-1} \circ b'_n &= \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} d_i d_j \\
&= \sum_j \left(\sum_{i<j} + \sum_{i \geq j} \right) (-1)^{i+j} d_i d_j \\
&= \sum_j \left(\sum_{i<j} (-1)^{i+j} d_i d_j + \sum_{k-1 \geq j} (-1)^{k-1+j} d_{k-1} d_j \right) \\
&= \sum_j \sum_{i<j} (-1)^{i+j} d_i d_j - \sum_j \sum_{n \geq k > j} (-1)^{k+j} d_{k-1} d_j \\
[(14)] &= \sum_{0 \leq i < j \leq n} (-1)^{i+j} d_{j-1} d_i - \sum_{0 \leq j < k \leq n} (-1)^{k+j} d_{k-1} d_j \\
&= 0.
\end{aligned}$$

This establishes a well-defined augmented chain complex of bimodules with mult as augmentation map. Thus, it is only left to show that this augmented bar complex is exact. For this, we introduce the operator $s = s_n := 1_A \otimes (-) \colon \mathrm{Bar}_{n-1}(A) \to \mathrm{Bar}_n(A)$ for $n \in \mathbb{N}_0$ that appends a unit entry and a separator on the left. In principle, the indices can be omitted and become clear from the context. However, in the following argument they are spelled out at least once. By construction, we have $d_0 s_n = \mathrm{Id}_{\mathrm{Bar}_{n-1}(A)}$ for the face map d_0 that drops the left most separator. Since it shifts the enumeration of the former separators, we also have $d_i s_n = s_{n-1} d_{i-1}$ for $1 \leq i \leq n$. Therefore,

$$b'_n s_n + s_{n-1} b'_{n-1} = \sum_{i=0}^{n} (-1)^i d_i s_n - \sum_{j=0}^{n-1} (-1)^j s_{n-1} d_j = d_0 s_n = \mathrm{Id}_{\mathrm{Bar}_{n-1}(A)}.$$

This shows that s defines a contracting homotopy. Thus, the augmented bar complex is indeed exact since any cycle $x \in \ker(b')$ is a boundary:

$$x = b' s(x) + s b'(x) = b' s(x) \in \mathrm{Im}(b').$$

□

Due to its generality, the bar resolution does not simplify the investigation of a specific algebra A. In practise, it rather serves as a starting point for the search of a possibly much shorter resolution. The actual advantage of having $\mathrm{Bar}_\bullet$ as some kind of default resolution is that it provides a general framework for further chain complexes. In fact, both the Hochschild cocomplex from Definition 2.41 and the Hochschild complex later in Definition 5.4 are images of the Bar complex under the bimodule Hom-functor $\mathrm{Hom}_{A,A}(-,M)$ or the two-sided balanced tensoring functor $M \otimes_{A,A} (-)$, respectively.

Proposition 4.9 ([24, Example 15.15])**.** *Let A be a unital algebra and let M be an A-bimodule. Then the image of the bar complex under the contravariant Hom-functor $\mathrm{Hom}_{A,A}(-,M)$ yields the Hochschild cocomplex*

$$(\mathrm{Hom}_{A,A}(\mathrm{Bar}_\bullet, M), (b')^*) \cong (\mathrm{Mulin}(A^\bullet, M), \delta).$$

In particular, it shows the claim $\delta^2 = 0$ for unital A that was left unproven in Definition 2.41. Since the proof of $(b')^2 = 0$ did not make use of unitality, it also proves $\delta^2 = 0$.

Proof. The Hom-functor maps $b' \colon \mathrm{Bar}_{n+1}(A) \to \mathrm{Bar}_n(A)$ to the operator

$$(b')^* = \sum_{j=0}^{n+1} (-1)^j \, (\cdot) \circ d_j \colon \mathrm{Hom}_{A,A}(\mathrm{Bar}_n(A), M) \to \mathrm{Hom}_{A,A}(\mathrm{Bar}_{n+1}(A), M)$$

that precomposes with it and clearly inherits $(b')^{*2} = 0$. Note that any bimodule homomorphism that starts from a free A-bimodule $A \otimes X \otimes A$ over a vector space X is pinned down by its values on $1 \otimes x \otimes 1$ for $x \in X$. So they are in bijection with linear maps starting in X. Applied to $X = A^{\otimes n}$, this leads to

$$\begin{aligned} \mathrm{Hom}_{A,A}(\mathrm{Bar}_n(A), M) &\cong \mathrm{Hom}(A^{\otimes n}, M) \\ F &\mapsto F(1 \otimes (\cdot) \otimes 1) \\ \mathrm{Id}_A \otimes \varphi \otimes \mathrm{Id}_A &\leftarrow\!\shortmid \varphi. \end{aligned}$$

By the universal property of the tensor product, the latter is in natural bijection with the n-linear maps from A^n to M as well. Under these bijections $(b')^*$ takes the promising shape of an operator

$$\tilde{b} \colon \mathrm{Mulin}(A^n, M) \to \mathrm{Mulin}(A^{n+1}, M).$$

Concretely, for any $F \in \mathrm{Hom}_{A,A}(\mathrm{Bar}_n(A), M)$ and any pure tensor $x = a_1 \otimes x^{(r)} \otimes a_{n+1}$ we have

$$
\begin{aligned}
(b')^*F(1\otimes x\otimes 1) &= \sum_{j=0}^{n+1}(-1)^j F d_j(1\otimes x\otimes 1)\\
&= F(x\otimes 1) + \sum_{j=1}^{n}(-1)^j F(1\otimes d_{j-1}(x)\otimes 1) + (-1)^{n+1}F(1\otimes x)\\
&= a_1 F(1\otimes x^{(r)}\otimes a_{n+1}\otimes 1) + \sum_{j=1}^{n}(-1)^j F(1\otimes d_{j-1}(x)\otimes 1)\\
&\quad + (-1)^{n+1}F(1\otimes a_1\otimes x^{(r)}\otimes 1)a_{n+1}.
\end{aligned}
$$

This computation allows to directly read off the corresponding linear map given by

$$
x \quad\mapsto\quad a_1\varphi(x^{(r)}\otimes a_{n+1}) + \sum_{j=1}^{n}(-1)^j\varphi(d_{j-1}(x)) + (-1)^{n+1}\varphi(a_1\otimes x^{(r)})a_{n+1}.
$$

From here, the action of the boundary map $\tilde{b}$ is effectively obtained by replacing all $\otimes$ separators by commas. By Definition 2.41 we therefore have the desired equality $\tilde{b} = \delta_n$. This shows the claim. □

This alternative interpretation of the Hochschild cocomplex helps to link quasi-free algebras and those with a projective reduced bimodule of noncommutative 1-forms. In fact, the corresponding lifting properties turn out to be equivalent, but for our purposes we only need one implication for now.

Theorem 4.10 ([24, Thm. 20.13]). *Let A be a unital and quasi-free algebra. Then $\overline{\Omega^1}(A)$ is a projective A-bimodule.*

Proof. The n-th boundary map of the bar complex surjects onto its image and since the bar complex has vanishing homology, it coincides with the kernel of the subsequent boundary map:

$$
\mathrm{Im}(b'_n) = \ker(b'_{n-1}).
$$

So if we restrict the range to this image, we obtain surjective bimodule maps with

$$
F_n\colon \mathrm{Bar}_n(A) \to \ker(b'_{n-1}), \quad F_n\circ b'_{n+1} = 0.
$$

Under the bijections in Proposition 4.9, they translate to n-linear maps with

$$\varphi_n \colon A^n \to \ker(b'_{n-1}), \quad \delta_n \varphi_n = 0.$$

Especially for $n = 2$, this shows that φ_2 is a Hochschild 2-cochain. Furthermore, since A is quasi-free by assumption, we also know $HH^2(A, \ker(b'_1)) = 0$ by Corollary 2.47. Hence, it is even a Hochschild 2-coboundary and there is a linear map $\psi \colon A \to \ker(b'_1)$ with $\varphi_2 = \delta_1 \psi$. Again by Proposition 4.9, this translates to the existence of a bimodule map with

$$\Psi \colon \mathrm{Bar}_1(A) \to \ker(b'_1), \quad \Psi \circ b'_2 = F_2.$$

For any $x \in \ker(b'_1)$ and any preimage x' with $F_2(x') = b'_2(x') = x$, we observe

$$\Psi(x) = \Psi(b'_2(x')) = F_2(x') = x.$$

Therefore, Ψ^2 acts the same as Ψ itself and we may view Ψ as a projection onto $\ker(b'_1)$. As a result, its complementary projection $\mathrm{Id}_{\mathrm{Bar}_1(A)} - \Psi$ establishes a bimodule isomorphism

$$(\mathrm{Id}_{\mathrm{Bar}_1(A)} - \Psi)(\mathrm{Bar}_1(A)) \cong \mathrm{Im}(b'_1) = \ker(b'_0) = \overline{\Omega^1}(A).$$

Thus, $\overline{\Omega^1}(A)$ materialises as a direct summand of the free bimodule $\mathrm{Bar}_1(A) = A \otimes A \otimes A$, which renders it projective by the argument in Proposition 2.19. This shows the claim. □

In the special case of a unital quasi-free algebra, the multiplication map alone is therefore already sufficient to build a significantly shorter projective bimodule resolution (12). If the quasi-free algebra is non-unital, though, then we may still pass to its unitalisation and obtain the sequence in (13) instead.

Corollary 4.11. *Let B be a generalised path algebra as in Definition 3.15. Let $L = \mathrm{Cohn}(B)$ be its associated relative Leavitt path algebra as in Theorem 3.16. Then the exact sequences*

$$\begin{aligned} 0 \to \Omega^1(B) \to B^+ \otimes B^+ \to B^+ \to 0, \\ 0 \to \Omega^1(L) \to L^+ \otimes L^+ \to L^+ \to 0 \end{aligned}$$

are projective bimodule resolutions of B^+ and L^+, respectively.

Proof. By Proposition 3.17 and Corollary 3.18 both B and L are quasi-free. Since k is also quasi-free by Proposition 2.49, this carries over to the unitalisations as well because the summandwise algebra splits can be combined to the desired split homomorphism. Hence, Theorem 4.10 shows that their bimodules of noncommutative forms are projective. Furthermore, also the tensor product bimodules are both free modules over the vector space $X = k$ and hence projective. This yields the claim. □

4.3 Computation of the reduced bimodule of noncommutative forms

While the bimodule of noncommutative forms arises as the kernel of the multiplication map for the unitalised algebra A^+, the reduced version is based on the multiplication map for A itself. In the non-unital case, this setup might not be good enough since $A \otimes A$ is not necessarily projective. Our algebras of interest, however, all have the advantage to have enough idempotents. In the following, we aim to show that this weaker form of unitality is still enough to establish a projective bimodule resolution without the detour involving unitalisations.

Lemma 4.12. *Let A be an algebra with enough idempotents. Then $A \otimes A$ is a projective A-bimodule.*

Proof. The orthogonal idempotents lead to a direct sum construction in Corollary 2.20 and establish that both one-sided modules ${}_AA$ and A_A are projective.
Furthermore, the adjoint interplay of tensor-functors and Hom-functors allows to reinterpret

$$\mathrm{Hom}_{A\otimes A}(A \otimes A, -) \cong \mathrm{Hom}_A(A, \mathrm{Hom}_A(A, -))$$

as a composition of two exact functors and thereby shows that both one-sided modules ${}_{(A\otimes A)}A \otimes A$ and $A \otimes A_{(A\otimes A)}$ are projective. Finally, as in Lemma 2.9, the existence of local units enables to construct one-sided sections for the multiplication map

$$\mathrm{mult}\colon A \otimes A \to A.$$

These sections transport the projectivity statements for $A \otimes A$ back to the A-module level. □

Corollary 4.13. *Let B be a generalised path algebra and let $L = \mathrm{Cohn}(B)$ be its associated relative Leavitt path algebra. Then both $B \otimes B$ and $L \otimes L$ are projective bimodules.*

The last result shows that (12) is still a reasonable candidate sequence. The goal for the rest of this section is to compute its kernel, that is, the reduced bimodule of noncommutative forms $\overline{\Omega^1}$ both for a generalised path algebra B and for its associated relative Leavitt path algebra L. As one might hope, the fact that L is a Cohn localisation allows to reduce the computational effort to the unlocalised algebra B.

Theorem 4.14. *Let A be an algebra with local units. Then for any Cohn localisation* $\mathrm{Cohn}(A)$ *both bimodule versions of noncommutative forms are spanned by the one for the unlocalised algebra A:*

$$\Omega^1(\mathrm{Cohn}(A)) = \mathrm{Cohn}(A)\Omega^1(A)\mathrm{Cohn}(A),$$
$$\overline{\Omega^1}(\mathrm{Cohn}(A)) = \mathrm{Cohn}(A)\overline{\Omega^1}(A)\mathrm{Cohn}(A).$$

Proof. Label the set of localisation maps by I. For all $i \in I$ let $n_i \in \mathbb{N}$ and $q^{(i)}, p^{(i)} \in \mathrm{Idem}(M_{n_i}(A))$ be such that the i-th localisation map is given by matrix multiplication with $Y^{(i)} \in q^{(i)} M_{n_i}(A) p^{(i)}$.
Then the Cohn localisation is obtained by adjoining universal matrix entries that form an inverse matrix multiplication map $X^{(i)} \in p^{(i)} M_{n_i}(\mathrm{Cohn}(A)) q^{(i)}$. Its relations are encoded by $X^{(i)} Y^{(i)} = p^{(i)}$ and $Y^{(i)} X^{(i)} = q^{(i)}$. The idea is to apply the universal derivation $d\colon \mathrm{Cohn}(A) \to d(\mathrm{Cohn}(A)) \subseteq \Omega^1(\mathrm{Cohn}(A))$ entrywise to the defining matrix relations. This gives the linear map

$$d_*\colon M_n(\mathrm{Cohn}(A)) \to M_n(d(\mathrm{Cohn}(A)))$$

between k-vector spaces. Note that $M_n(d(\mathrm{Cohn}(A)))$ is not an algebra itself, but rather a subvector space of $M_n(\Omega^1(\mathrm{Cohn}(A)))$, on which $M_n(\mathrm{Cohn}(A))$ acts via matrix multiplication. With respect to this structure, d_* is still a derivation due to entrywise application of the Leibniz rule

$$\begin{aligned}
d_*(ab) &= \left(\sum_{j\leq n} d(a_{ij}b_{jk})\right)_{i,k\leq n} \\
&= \left(\sum_{j\leq n} d(a_{ij})b_{jk} + a_{ij}d(b_{jk})\right)_{i,k\leq n} \\
&= \left(\sum_{j\leq n} d(a_{ij})b_{jk}\right)_{i,k\leq n} + \left(\sum_{j\leq n} a_{ij}d(b_{jk})\right)_{i,k\leq n} \\
&= d_*(a)b + ad_*(b)
\end{aligned}$$

for $a, b \in M_n(\mathrm{Cohn}(A))$. Hence, if we drop the labelling $(.)^{(i)}$ for notational convenience in the environment above, the application of d_* turns the defining relations into

$$\begin{aligned}
&(1) \quad d_*(X) = d_*(Xq) = d_*(X)q + Xd_*(q), \\
&(2) \quad d_*(p) = d_*(XY) = d_*(X)Y + Xd_*(Y), \\
&(3) \quad d_*(X) = d_*(pX) = d_*(p)X + pd_*(X), \\
&(4) \quad d_*(q) = d_*(YX) = d_*(Y)X + Yd_*(X).
\end{aligned}$$

Now, multiplication of (2) with $() \cdot X$ in combination with $q = YX$ yields

$$d_*(p)X = d_*(X)q + Xd_*(Y)X.$$

This allows for a formulation of $d_*(X)q$ that does not depend on $d_*(X)$. Plugging this into (1) shows

$$d_*(X) = d_*(p)X - Xd_*(Y)X + Xd_*(q).$$

Alternatively, one could also use $X \cdot (4)$, $p = XY$ and (3) to arrive at the same expression. It says that $d_*(X)$ lies in the span

$$M_n(\mathrm{Cohn}(A)^+)M_n(d(A))M_n(\mathrm{Cohn}(A)^+) = M_n(\mathrm{Cohn}(A)^+d(A)\mathrm{Cohn}(A)^+).$$

For the adjoined matrix entries in $X = (x_{ij})$, we therefore have

$$d(x_{ij}) \in \mathrm{Cohn}(A)^+d(A)\mathrm{Cohn}(A)^+.$$

Since this holds for all constructed inverse matrices and since Lemma 4.6 says that span considerations reduce to generators, we have that

$$d(\mathrm{Cohn}(A)) \subseteq \mathrm{Cohn}(A)^+ d(A)\mathrm{Cohn}(A)^+.$$

Now, again by Lemma 4.6, it follows

$$\begin{aligned}\Omega^1(\mathrm{Cohn}(A)) &= \mathrm{Cohn}(A)^+ d(\mathrm{Cohn}(A))\mathrm{Cohn}(A)^+ \\ &\subseteq \mathrm{Cohn}(A)^+ d(A)\mathrm{Cohn}(A)^+ \\ &= \mathrm{Cohn}(A)^+ A^+ d(A) A^+ \mathrm{Cohn}(A)^+ \\ &= \mathrm{Cohn}(A)^+ \Omega^1(A)\mathrm{Cohn}(A)^+.\end{aligned}$$

The relations $X^{(i)} \in p^{(i)} M_{n_i}(\mathrm{Cohn}(A))q^{(i)}$ for all $i \in I$ even ensure that the adjoined generators are in $A\mathrm{Cohn}(A)A$. Hence, $\mathrm{Cohn}(A)$ is also a non-degenerate A-bimodule. This allows for an analogous argument in the reduced case

$$\begin{aligned}\overline{\Omega^1}(\mathrm{Cohn}(A)) &= \mathrm{Cohn}(A) d(\mathrm{Cohn}(A))\mathrm{Cohn}(A) \\ &\subseteq \mathrm{Cohn}(A) d(A)\mathrm{Cohn}(A) \\ &= \mathrm{Cohn}(A) A d(A) A \mathrm{Cohn}(A) \\ &= \mathrm{Cohn}(A)\overline{\Omega^1}(A)\mathrm{Cohn}(A).\end{aligned}$$

Both inverse containments now follow from the trivial observation $\Omega^1(A) \subseteq \Omega^1(\mathrm{Cohn}(A))$ or $\overline{\Omega^1}(A) \subseteq \overline{\Omega^1}(\mathrm{Cohn}(A))$, respectively. This completes the proof. □

Proposition 4.15 ([22]). *Let B be a generalised path algebra. Then its reduced bimodule of noncommutative forms is smooth and satisfies*

$$\overline{\Omega^1}(B) = \bigoplus_{v,w \in E^0} v\overline{\Omega^1}(B)w = \bigoplus_{v,w \in E^0} v\Omega^1(B)w.$$

Any element $\omega \in \overline{\Omega^1}(B)$ is pinned down by the finitely supported sum

$$\omega = \sum_{v,w \in E^0} v\omega w.$$

Furthermore, B is generated by $E^0 \cup E^1 \cup S$, that is, by vertices, edges, and source projec-

tions. Consequently,

$$\overline{\Omega^1}(B) = Bd(B)B = Bd(E^0 \cup E^1 \cup S)B = \bigoplus_{v,w \in E^0} Bvd(E^0 \cup E^1 \cup S)wB.$$

Proof. Since B has enough idempotents, we know from Lemma 2.9 that a B-bimodule M is smooth if and only if any element $m \in M$ is invariant under an idempotent in B from either side. Expressed with vertices, this boils down to the property that M is finitely generated over its own vertex corners

$$M = \bigoplus_{v,w \in E^0} vMw.$$

Here, any representation of $m \in M$ with $n \in \mathbb{N}$ summands

$$m = \sum_{(v_j,w_j) \in E^0 \times E^0,\, j \leq n} v_j m_{v_j,w_j} w_j$$

gives by orthogonality that $v_j m w_j = v_j m_{v_j,w_j} w_j$ for all $j \leq n$, while the matrix coefficients for the remaining vertex pairs $(v,w) \neq (v_j, w_j)$ have to vanish. Hence, without loss of generality we can phrase an element $m \in M$ in a smooth B-bimodule as

$$m = \sum_{v,w \in E^0} vmw,$$

where the sum is finitely supported. Since Lemma 2.9 also implies that B is smooth as a bimodule over itself, all B-spans are smooth bimodules, too. In particular, $B \otimes B = B(B \otimes B)B$ and by Lemma 4.6 also $\overline{\Omega^1}(B) = B\Omega^1(B)B = Bd(B)B$ are smooth bimodules. The discussion for $M = \overline{\Omega^1}(B)$ above now implies the first part, while the second part is a consequence of Lemma 4.6. $\square$

In the study of $\overline{\Omega^1}(B)$, Proposition 4.15 allows to focus on noncommutative forms $vd(x)w$ for $x \in E^0 \cup E^1 \cup S$ and $v, w \in E^0$. Ideally, we manage to identify explicit direct summands of this shape $Bvd(x)wB \cong Bv \otimes wB$. Since they are corner modules of $B \otimes B$ they are known to be projective by Lemma 2.19 and Lemma 4.12. As always, the effect of d on vertices should be studied first before we turn to the whole generalised path algebra B.

Proposition 4.16 ([22]). *Let E be a quiver and denote the algebra generated by its vertices* *as $k[E^0] = \bigoplus_{v\in E^0} vk$. Then its reduced bimodule of noncommutative forms admits the following direct sum decomposition*

$$\overline{\Omega^1}(k[E^0]) = \bigoplus_{v\neq w\in E^0} k[E^0]vd(v)wk[E^0] = \bigoplus_{v\neq w\in E^0} (vd(v)w)k.$$

In particular, in the case of finitely many orthogonal idempotents $|E^0| = n$, this includes the discussion for the free commutative algebra k^n. Reduplication $v \mapsto v\otimes v$ provides a split for the multiplication map while all other elementary tensors $v\otimes w$ for $v \neq w$ belong to the kernel and generate the $n(n-1)$ dimensional vector space $\overline{\Omega^1}(k^n)$.

$$\bigoplus_{v\neq w\in E^0} (vd(v)w)k \hookrightarrow \bigoplus_{v,w\in E^0} (v\otimes w)k \twoheadrightarrow \bigoplus_{v\in E^0} vk$$

$$v\otimes w \mapsto \delta_{v,w}v$$

$$v\otimes v \leftarrow\!\shortmid v.$$

Proof. Since two different vertices $v \neq w \in E^0$ are orthogonal idempotents by (V), the Leibniz rule gives

$$0 = d(vw) = vd(w) + d(v)w.$$

From here, the multiplication with another $x \perp v$ implies $xd(v)w = 0$, while the multiplication by $v\cdot()\cdot w$ gives $vd(v)w = -vd(w)w$. Furthermore,

$$d(v) = vd(v) + d(v)v \iff (1-v)d(v) = d(v)v \iff d(v)(1-v) = vd(v).$$

In this context, $1 \in k[E^0]^+$ is meant to be the identity operator. If we additionally multiply by v or $1-v$, this implies

$$vd(v)v = 0, \quad (1-v)d(v)(1-v) = 0. \tag{15}$$

In total, this shows that the subset $(vd(v)w)_{v\neq w\in E^0}$ of the spanning elements

$$\{vd(x)w \mid v,x,w \in E^0\}$$

in Proposition 4.15 already generates the whole kernel and by orthogonality it does not matter whether one looks at the $k[E^0]$- or k-linear span. Since $E^0\otimes E^0$ forms a basis of

$k[E^0] \otimes k[E^0]$, these generators are linearly independent because

$$vd(v)w = (v \otimes v - v \otimes 1)w = -v \otimes w.$$

This shows that the sums are direct and gives the claim.
Spelled out in the special case of k^n with entrywise multiplication, we use the standard basis $E^0 := \{p_j = (\delta_{i,j})_{i \leq n} \mid j \leq n\} \subseteq k^n$ as vertices. They clearly provide a set of pairwise orthogonal idempotents. Plugged in, the above split extension now reads as:

$$\left(\overline{\Omega^1}(k^n) = \bigoplus_{i \neq j \leq n} (p_i \otimes p_j)k\right) \hookrightarrow \left(k^n \otimes k^n = \bigoplus_{i,j \leq n} (p_i \otimes p_j)k\right) \twoheadrightarrow \left(k^n = \bigoplus_{i \leq n} p_i k\right)$$
$$p_i \otimes p_j \mapsto \delta_{i,j} p_i,$$
$$p_i \otimes p_i \leftarrow\!\shortmid p_i. \qquad \square$$

When it comes to source projections, we have already used an artificial enumeration of the orthogonal idempotents $(p_e)_{e \in s^{-1}(v)}$ for the unchosen vertices $v \in (\mathrm{Reg}(E) \setminus X) \cup \mathrm{Inf}(E)$ in the proof of Proposition 3.17 to address them iteratively. The enumeration amounts to a bijective map between the countable index sets $\varphi\colon \mathbb{N}_{\leq |s^{-1}(v)|} \to s^{-1}(v)$ and induces a total order $\leq$ on this fibre in the obvious way:

$$e = \varphi(n) \leq \varphi(m) = f \iff n \leq m \quad \text{for} \quad e, f \in s^{-1}(v);\, n, m \in \mathbb{N}_{\leq |s^{-1}(v)|}\,.$$

Given an edge $e \in s^{-1}(E^0 \setminus X)$ it allows for the shorthand notation $\sum_{f \leq e}$, which means that we sum over all edges f with $s(f) = s(e)$ that are labelled with a natural number less than or equal to the label of e itself.

Definition 4.17 ([2, p. 17]). Let B be a generalised path algebra. Fix a bijection $\varphi\colon \mathbb{N}_{\leq |s^{-1}(v)|} \to s^{-1}(v)$ for every vertex $v \notin \mathrm{Sink}(E)$ with a non-empty fibre. If it is an *unchosen vertex* $v \in (\mathrm{Reg}(E) \setminus X) \cup \mathrm{Inf}(E)$, we use the induced total order on the fibre to define the so called *residual projection* for $e \in s^{-1}(E^0 \setminus X)$ as

$$q_e := s(e) - \sum_{f \leq e} p_f \in B.$$

Lemma 4.18. *The residual projections are idempotents in B with $q_e \perp p_e$ and $s(e)q_e = q_e = q_e s(e)$ for all edges $e \in s^{-1}(E^0 \setminus X)$. Remember that the second property is captured by the notation $q_e \leq s(e)$ for idempotents that was introduced in Definition 2.7.*

In fact, taking residual projections reverses the chosen order on the fibres in the sense that it carries $\le$-relations between edges to the opposite $\le$-relation on the level of idempotents:

$$e \le f \iff q_f \le q_e.$$

The common source vertex $s(e) = s(f)$ also fits into this correspondence and can be interpreted both as the minimal element in the sense of the zeroth edge for the fibre ordering and as the maximal element of the idempotent chain via $q_{s(e)} := s(e)$.

Proof. Since the source projections $(p_f)_{s(f)=s(e)}$ are orthogonal idempotents with $p_f \le s(e)$, the claims in the first paragraph immediately follow from 2.7. Now, consider two edges in a common fibre $e \le f$. Then

$$\begin{aligned}
q_e q_f &= s(e)s(f) - s(e)\sum_{g\le f} p_g - \sum_{g \le e} p_g s(f) + \sum_{g\le e, h\le f} p_g p_h, \\
&= s(f) - \sum_{e<g\le f} p_g - 2\sum_{g\le e} p_g + \sum_{g\le e} p_g, \\
&= s(f) - \sum_{g \le f} p_g, \\
&= q_f.
\end{aligned}$$

Note that in the same way we obtain $q_f q_e = q_f$ and hence $q_f \le q_e$. Since the above computation only relies on $f = \max(e, f)$, it even shows $q_e q_f = q_f q_e = q_{\max(e,f)}$ for general edges in a common fibre $s(e) = s(f)$.
Conversely, if we know $q_f \le q_e \le s(e)$ for two edges $e, f \in s^{-1}(E^0 \setminus X)$, then they have to be in the same fibre $s(e) = s(f)$ by vertex orthogonality and the above computation shows $q_f = q_{\max(e,f)}$ by assumption. In order to have the same summands in the definition of the residual projections, we need to have $f = \max(e, f)$, that is, $e \le f$. This shows the claimed equivalence. The final statement now follows from $q_e \le s(e)$. □

Remark 4.19. Note that if we enumerate the finite fibre for $v \in \mathrm{Reg}(E)$, then we also highlight a *maximal edge* e_v. If v is unchosen, then the associated minimal residual projection

$$q_{e_v} = v - \sum_{e \in s^{-1}(v)} p_e$$

measures the failure of the unimposed $(CK2)$-relation. In particular, with an eye on

Corollary 3.8, both Cohn localisations for the choices $\emptyset$ and X are related via

$$L_k^X(E) = \frac{C(E)}{C(E)\{q_{e_x} \mid x \in X\}C(E)}.$$

Back to the study of $\overline{\Omega^1}(B)$, the notion of residual projections also allows to identify further direct summands of this bimodule.

Proposition 4.20 ([22])**.** *Let v be an unchosen vertex as in Definition 4.17 and let $\leq$ be a fixed order on the fibre $s^{-1}(v)$. For any edge $f \in s^{-1}(v)$ let*

$$A_f := k[v, (p_e)_{e \leq f}]$$

be the algebra generated by v and the finite number of source projections associated to edges $\leq f$. Then its reduced bimodule of noncommutative forms admits the following direct sum decomposition

$$\overline{\Omega^1}(A_f) = \bigoplus_{e \leq f} (A_f p_e d(p_e) q_e A_f \oplus A_f q_e d(p_e) p_e A_f)\,. \tag{16}$$

If the algebra generated by v and all of its source projections is denoted by

$$A_\infty := k[v, (p_e)_{e \in s^{-1}(v)}],$$

then it also holds

$$\overline{\Omega^1}(A_\infty) = \bigoplus_{e \in s^{-1}(v)} (A_\infty p_e d(p_e) q_e A_\infty \oplus A_\infty q_e d(p_e) p_e A_\infty)\,.$$

Proof. First of all, $s^{-1}(v)$ merely serves as a countable index set and for notational convenience during this proof we may directly use the enumeration $\{p_n \mid n \in \mathbb{N}_{\leq |s^{-1}(v)|}\}$ and corresponding A_n instead. Put this way, we have to show

$$\overline{\Omega^1}(A_n) = \bigoplus_{j \leq n} (A_n p_j d(p_j) q_j A_n \oplus A_n q_j d(p_j) p_j A_n) \tag{17}$$

for all $n \in \mathbb{N}_{\leq |s^{-1}(v)|}$. If the unchosen vertex is regular with $|s^{-1}(v)| < \infty$, then $A_\infty = A_{|s^1(v)|}$ is also covered by (17). So, without loss of generality, we may assume that $v \in \mathrm{Inf}(E)$ is an infinite emitter from now on and aim for a proof of (17) by induction. The residual projection $q_n \in A_n$ allows to rewrite the unit element

$$v = \sum_{j \le n} p_j + q_n$$

as a decomposition of orthogonal idempotents and therefore renders A_n as the k-algebra generated by these $n+1$ pairwise orthogonal idempotents $\{p_j, q_n \mid j \le n\}$. This setup is already investigated in Proposition 4.16 and yields the $(n+1)^2 - (n+1) = (n+1)n$ summands

$$\overline{\Omega^1}(A_n) = \bigoplus_{i \neq j \le n} A_n p_i d(p_i) p_j A_n \oplus \bigoplus_{i \le n} (A_n p_i d(p_i) q_n A_n \oplus A_n q_n d(q_n) p_i A_n) .$$

Here, the last summand $A_n q_n d(q_n) p_i A_n$ can be replaced by $A_n q_n d(p_i) p_i A_n$ since $q_n \perp p_i$ implies $q_n d(q_n) p_i = -q_n d(p_i) p_i$. In particular, Proposition 4.16 shows the claim (17) for A_1:

$$\overline{\Omega^1}(A_1) = A_1 p_1 d(p_1) q_1 A_1 \oplus A_1 q_1 d(p_1) p_1 A_1 .$$

If we pass from any A_n, for which (17) holds, to A_{n+1}, however, the residual projection q_n is further split into two individual orthogonal idempotents $q_n = q_{n+1} \oplus p_{n+1}$ that both lie in $A_{n+1} \setminus A_n$. Hence, with elements from A_{n+1} at hand, we have that

$$\begin{aligned} \overline{\Omega^1}(A_n) = & \bigoplus_{i \neq j \le n} A_n p_i d(p_i) p_j A_n \oplus \bigoplus_{i \le n} A_n p_i d(p_i) [q_{n+1} + p_{n+1}] A_n \\ & \oplus \bigoplus_{i \le n} A_n [q_{n+1} + p_{n+1}] d(p_i) p_i A_n . \end{aligned}$$

In combination with $A_n A_{n+1} = A_{n+1}$, this causes the following shape of the A_{n+1}-span in the usual product notation:

$$\begin{aligned} A_{n+1} \overline{\Omega^1}(A_n) A_{n+1} = & \bigoplus_{i \neq j \le n} A_{n+1} p_i d(p_i) p_j A_{n+1} \\ & \oplus \bigoplus_{i \le n} (A_{n+1} p_i d(p_i) q_{n+1} A_{n+1} \oplus A_{n+1} p_i d(p_i) p_{n+1} A_{n+1}) \\ & \oplus \bigoplus_{i \le n} (A_{n+1} q_{n+1} d(p_i) p_i A_{n+1} \oplus A_{n+1} p_{n+1} d(p_i) p_i A_{n+1}) . \end{aligned}$$

At this stage, we can again use the equations $q_{n+1} d(p_i) p_i = -q_{n+1} d(q_{n+1}) p_i$ and likewise $p_{n+1} d(p_i) p_i = -p_{n+1} d(p_{n+1}) p_i$ to compare these summands with the result in Proposition

4.16 for A_{n+1}. In fact, only two direct summands

$$S_1 := A_{n+1}q_{n+1}d(p_{n+1})p_{n+1}A_{n+1}, \quad S_2 := A_{n+1}p_{n+1}d(p_{n+1})q_{n+1}A_{n+1}$$

are missing and by induction hypothesis we obtain

$$\begin{aligned}\overline{\Omega^1}(A_{n+1}) &= A_{n+1}\overline{\Omega^1}(A_n)A_{n+1} \oplus S_1 \oplus S_2 \\ &= \bigoplus_{j\leq n} (A_{n+1}p_jd(p_j)q_jA_{n+1} \oplus A_{n+1}q_jd(p_j)p_jA_{n+1}) \oplus S_1 \oplus S_2 \\ &= \bigoplus_{j\leq n+1} (A_{n+1}p_jd(p_j)q_jA_{n+1} \oplus A_{n+1}q_jd(p_j)p_jA_{n+1}) .\end{aligned}$$

This completes the proof of (17). As in Example 2.10, the algebra A_∞ is the union or, in other words, the inductive limit of the system formed by the algebras $(A_n)_{n\in\mathbb{N}}$ and their canonical inclusions:

$$A_\infty = \bigcup_{n\in\mathbb{N}} A_n.$$

Indeed, every element is a finitely supported linear combination of generators and thus belongs to some A_n if n denotes the maximal index with a non-vanishing coefficient for p_n. Since (17) gives a compatible direct sum decomposition for increasing n, this result also passes on to the direct limit, as claimed:

$$\begin{aligned}\overline{\Omega^1}(A_\infty) &= \bigcup_{m\in\mathbb{N}} A_\infty\overline{\Omega^1}(A_m)A_\infty \\ &= \bigcup_{m\in\mathbb{N}}\bigoplus_{n\leq m} (A_\infty p_nd(p_n)q_nA_\infty \oplus A_\infty q_nd(p_n)p_nA_\infty) \\ &= \bigoplus_{n\in\mathbb{N}} (A_\infty p_nd(p_n)q_nA_\infty \oplus A_\infty q_nd(p_n)p_nA_\infty) .\end{aligned}$$ □

Theorem 4.21 ([22]). *Let B be a generalised path algebra with residual projections as in Definition 4.17. Then the reduced bimodule of noncommutative forms is a projective bimodule and admits the following direct sum decomposition*

$$\begin{aligned}\overline{\Omega^1}(B) = &\bigoplus_{v\neq w\in E^0} Bvd(v)wB \oplus \bigoplus_{e\in s^{-1}(E^0\setminus X)} (Bp_ed(p_e)q_eB \oplus Bq_ed(p_e)p_eB) \\ &\oplus \bigoplus_{e\in s^{-1}(X)} Bs(e)d(e)r(e)B \oplus \bigoplus_{e\in s^{-1}(E^0\setminus X)} Bp_ed(e)r(e)B.\end{aligned}$$

Proof. It is clear that all summands on the right hand side are contained in $\overline{\Omega^1}(B)$. By Lemma 2.19 and Lemma 4.12 we also know that all these summands are indeed projective. So it is left to show that the sums are direct and already span the whole kernel. By Proposition 4.15 it suffices to consider elements of the shape $vd(x)w$ with $v, w \in E^0$ and $x \in E^0 \cup E^1 \cup S$. For $x \in E^0$, the discussion in Proposition 4.16 breaks things down to the first big summand

$$\bigoplus_{v \neq w \in E^0} Bvd(v)wB$$

that occurs in the claim, while $vd(v)w = -v \otimes w$ shows that it is a direct sum on its own. Next, for a source projection $x = p_e \in S$ with $e \in s^{-1}(E^0 \setminus X)$, the Leibniz rule yields

$$d(p_e) = d(s(e)p_e s(e)) = d(s(e))p_e + s(e)d(p_e)s(e) + p_e d(s(e)).$$

Thus, any $vd(x)w$ can be phrased in terms of the already identified summands and

$$s(e)d(p_e)s(e) \in Bs(e) \otimes s(e)B.$$

This portion with a common vertex next to the tensor sign is clearly independent from the first big vertex summand. Now, the discussion in Proposition 4.20 applies to the algebra generated by the unchosen vertex $v = s(e)$ and all associated source projections. Indeed, it shows that

$$s(e)d(p_e)s(e) \in \bigoplus_{f \in s^{-1}(v)} (Bp_f d(p_f)q_f B \oplus Bq_f d(p_f)p_f B)$$

holds and that all involved sums are direct. In case of a regular unchosen vertex, even the summands from Proposition 4.16 can be used instead. Combining all of these portions for unchosen vertices yields the second big summand

$$\bigoplus_{e \in s^{-1}(E^0 \setminus X)} (Bp_e d(p_e)q_e B \oplus Bq_e d(p_e)p_e B)$$

that occurs in the claim. It is a direct sum on its own, since all of the involved tensors $p_e d(p_e) q_e = -p_e \otimes q_e$ or $q_e d(p_e) p_e = q_e \otimes p_e$ are independent from their counterparts for an edge $e' \in s^{-1}(E^0 \setminus X)$ with $s(e) \neq s(e')$. Again, the two big summands for the vertices on the one hand and for the source projections on the other hand also have a trivial intersection since the latter only deals with path families that admit a common source vertex next to $\otimes$.

Finally, for $x = e \in E^1$ we observe that the Leibniz rule yields

$$d(e) = d(s(e)er(e)) = d(s(e))e + s(e)d(e)r(e) + ed(r(e)).$$

This allows to phrase any $vd(x)w$ in terms of the already identified summands and

$$s(e)d(e)r(e).$$

In case of $e \in s^{-1}(E^0 \setminus X)$, the source vertex above can even be replaced by the source projection $p_e \leq s(e)$. Hence, the edge part of d in $\overline{\Omega^1}(B)$ is fully covered by the the additional summands

$$\bigoplus_{e\in s^{-1}(X)} Bs(e)d(e)r(e)B \oplus \bigoplus_{e\in s^{-1}(E^0\setminus X)} Bp_e d(e)r(e)B.$$

They finally make up the rest of the claimed right hand side. They are direct on its own due to the absence of any further relation among the edges $e \in E^1$. In fact, they also have trivial intersection with the prior two big summands from the investigation of vertices and source projections due to the $\mathbb{Z}$-grading. There is a path length imbalance in $s(e)d(e)r(e) = s(e) \otimes e - e \otimes r(e)$ for $e \in s^{-1}(X)$ or also in $p_e d(e)r(e) = p_e \otimes e - e \otimes r(e)$ for $e \in s^{-1}(E^0 \setminus X)$ of the shape $(0,1) - (1,0)$ between the left and the right side of $\otimes$. This imbalance is sustained by non-trivial multiplication with $b \in B$ on either side and therefore only allows for the trivial intersection with the prior summands that are all of $(0,0)$-shape.
Putting the three major investigations for vertices, source projections and edges together, this yields the claim. $\square$

Theorem 4.22. *Let $L = L_k^X(E)$ be a relative Leavitt path algebra. Then we have*

$$\begin{aligned}\overline{\Omega^1}(L) = &\bigoplus_{v\neq w\in E^0} Lvd(v)wL \oplus \bigoplus_{e\in s^{-1}(E^0\setminus X)} (Lee^*d(ee^*)q_eL \oplus Lq_ed(ee^*)ee^*L)\\ &\oplus \bigoplus_{e\in s^{-1}(X)} Ls(e)d(e)r(e)L \oplus \bigoplus_{e\in s^{-1}(E^0\setminus X)} Lee^*d(e)r(e)L.\end{aligned}$$

Furthermore, the exact sequence

$$0 \to \overline{\Omega^1}(L) \to L \otimes L \to L \to 0$$

resulting from the multiplication map is a projective bimodule resolution of length 1.

Proof. By Theorem 4.21 and Theorem 4.14, $\overline{\Omega^1}(L)$ is of the displayed shape. Moreover, all involved summands are isomorphic to $Lx \otimes yL$ for certain $x, y \in \mathrm{Idem}(L)$ and indeed projective. □

5 Selected homology theories for Leavitt path algebras

For a general algebra, the underlying vector space or, in other words, the k-module structure is usually relatively easy to understand. In fact, most of the complexity and diversity of algebras as mathematical objects is encoded in the multiplicative structure. With the free tensor algebra of the underlying vector space and the corresponding free algebra extension we already met one tool to capture the multiplicative peculiarities of an algebra A. They are described by implementing relations on the free tensor multiplication that span the kernel JA of the free extension. This is a somewhat intrinsic approach, which strips off the algebra structure initially, creates the free algebra structure that distinguishes all formal tensor words in A, and finally remembers all tensor word combinations that vanish once we replace every formal tensor $\otimes$ by the old multiplication.
Another approach to capture the multiplication in A is to use the regular representations

$$\begin{aligned}\lambda\colon A \to \mathrm{End}(A), &\quad a \mapsto a \cdot (),\\ \rho\colon A \to \mathrm{End}(A), &\quad a \mapsto () \cdot a.\end{aligned}$$

They allow to study the behaviour of an element $a \in A$ by its multiplication operators from the left or from the right, respectively. It corresponds to the external interpretation of A as a bimodule over itself. In fact, the idea to learn more about the algebra structure of A by its representations as multiplication operators on general vector spaces, not just on A itself, corresponds to the study of A-bimodules M. Since the category of A-bimodules and bimodule homomorphisms suits for homological algebra, we can benefit from its conceptually much more general theory that is designed for abelian categories at this point. For instance, homological algebra deals with concepts like chain complexes, projective bimodule resolutions or derived functors in a minimalistic setup. Nevertheless, the theory can also be spelled out in our special bimodule environment over an algebra on a lower layer of abstraction. Among other things, this also leads to the notion of Hochschild homology and cohomology. They turn out to be of central importance to phrase structural information about A and also lead to more distinct descriptions of algebra types apart from just free ones. In this way, we come back to quasi-free algebras and use their projective bimodule resolution of length one for homological computations.

5.1 Hochschild homology

If we apply these concepts to the Hochschild homology of a relative Leavitt path algebra L, this merely leaves to compute the kernel and the cokernel of a single boundary map $b\colon \Omega^1(L)_\# \to L$. For this task, we can benefit both from the direct sum decomposition of $\overline{\Omega^1}(L)$ and from the $\mathbb{Z}$-grading on L itself. In the end, both the subspace of 0-homogeneous elements L_0 and the closed path structure of E play major roles. Our route over b carries us to the same results that are already established for row-finite quivers. The main references for this other line of reasoning that is based on crossed products are [5], [6] and also [2, p. 253 ff.].

Proposition 5.1 ([18, p. 148])**.** *Let A be an algebra. Taking the two-sided balanced tensor product with A gives a right exact functor from the category of A-bimodules to the category of k-vector spaces.*

$$\begin{aligned} F\colon \mathrm{Mod}_{A,A} &\to \mathrm{Vect}_k \\ M &\mapsto A \otimes_{A,A} M \\ (\varphi\colon M \to N) &\mapsto \mathrm{Id}_A \otimes_{A,A} \varphi. \end{aligned}$$

For a self-induced algebra A and smooth bimodules M, it corresponds to taking commutator quotients

$$\begin{tikzcd} A \otimes_{A,A} M \arrow[r, "\mathrm{Id}_A \otimes_{A,A} \varphi"] \arrow[d, "\cong"] & A \otimes_{A,A} N \arrow[d, "\cong"] \\ M_\# \arrow[r, "\varphi_\#"] & N_\# \end{tikzcd}$$

with the induced map $\varphi_\#(m + [A,M]) := \varphi(m) + [A,N]$ that is well-defined because

$$\varphi([a,m]) = [a, \varphi(m)], \quad a \in A,\ m \in M.$$

Proof. F is a functor by construction of the tensor operation for bimodule homomorphisms. Given two bimodule homomorphisms $\varphi\colon M \to N$ and $\psi\colon N \to P$, functoriality reads as

$$\begin{aligned} &F(\psi) \circ F(\varphi) = (\mathrm{Id}_A \otimes_{A,A} \psi) \circ (\mathrm{Id}_A \otimes_{A,A} \varphi) = \mathrm{Id}_A \circ \mathrm{Id}_A \otimes_{A,A} \psi \circ \varphi = F(\psi \circ \phi), \\ &F(\mathrm{Id}_M) = \mathrm{Id}_A \otimes_{A,A} \mathrm{Id}_M = \mathrm{Id}_{A \otimes_{A,A} M} = \mathrm{Id}_{F(M)}. \end{aligned}$$

In the same fashion as for all tensoring functors it also right exact. Let

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0$$

be an exact sequence of bimodules. The claim is that F yields an exact sequence

$$F(M) \xrightarrow{F(\varphi)} F(N) \xrightarrow{F(\psi)} F(P) \longrightarrow 0\,. \tag{18}$$

Equivalently, we have to show that F preserves cokernels

$$P \cong \operatorname{coker}(\varphi) \implies F(P) \cong \operatorname{coker}(F(\varphi))$$

in order to get that (18) is isomorphic to the cokernel sequence

$$F(M) \xrightarrow{F(\varphi)} F(N) \xrightarrow{\pi} \operatorname{coker}(F(\varphi)) \longrightarrow 0\,,$$

which is exact by construction. Indeed, $F(\psi) \circ F(\varphi) = F(\psi \circ \varphi) = F(0) = 0$, so $F(\psi)$ factors through the cokernel of $F(\varphi)$ and we get a map

$$\psi' \colon \operatorname{coker}(F(\varphi)) \to F(P).$$

It satisfies $\psi' \circ \pi = F(\psi)$. Now, since ψ is surjective, any $p \in P$ has a preimage $n \in N$. The element $\pi(a \otimes_{A,A} n)$ only depends on $a \in A$ and $p \in P$, but not on the choice of the preimage because the difference of any two choices n, n' is in $\ker(\psi) = \varphi(M)$ such that $a \otimes_{A,A} (n - n')$ is always mapped to zero under π.
Moreover, the assignment $f(a,p) := \pi(a \otimes_{A,A} n)$ is clearly linear in a, but also in p since linearity of ψ ensures that the sum of preimages is a preimage of the sum of two elements in P. Even more, for $b, c \in A$ we have that $\psi(cnb) = c\psi(n)b$ and as a result

$$f(bac, p) = \pi(bac \otimes_{A,A} n) = \pi(a \otimes_{A,A} cnb) = f(a, cpb).$$

Thus, by the universal property of the two-sided balanced tensor product, the map f extends to a bimodule homomorphism $\omega \colon F(P) \to \operatorname{coker}(F(\varphi))$ with $\omega(a \otimes_{A,A} p) = \pi(a \otimes_{A,A}$

n). It provides a two-sided inverse of ψ' since

$$\omega \circ \psi'(\pi(a \otimes_{A,A} n)) = \omega \circ F(\psi)(a \otimes_{A,A} n) = \omega(a \otimes_{A,A} \psi(n)) = \pi(a \otimes_{A,A} n),$$
$$\psi' \circ \omega(a \otimes_{A,A} p) = \psi'(\pi(a \otimes_{A,A} n)) = F(\psi)(a \otimes_{A,A} n) = a \otimes_{A,A} p.$$

This completes the proof that F is right exact.
Finally, in the smooth case the isomorphism in Lemma 2.6 allows to reexpress $F(\varphi)$ as a map between commutator quotients and the diagram

$$\begin{tikzcd} A \otimes_{A,A} M \arrow[r, "\mathrm{Id}_A \otimes_{A,A} \varphi"] \arrow[d, "\cong"] & A \otimes_{A,A} N \arrow[d, "\cong"] \\ M_\# \arrow[r, dotted, "\varphi_\#"] & N_\# \end{tikzcd}$$

can be seen as the definition of $\varphi_\#$. Tracing this defining composition with the knowledge that φ maps commutators to commutators, it is explicitly given by $\varphi_\#(m + [A, M]) = \varphi(m) + [A, N]$. □

Remark 5.2 ([18, pp. 149, 389-391]). With an analogous procedure one can also prove that various types of tensoring functors are right exact. Only the argument for the construction of ω needs to be adapted to the respective universal property. For example, given a ring R and a right R-module M_R we could also consider the one-sided tensor product

$$F = M_R \otimes_R (-) \colon {}_R\mathrm{Mod} \to \mathrm{Ab}$$

that glues it with a left module ${}_RN$ along the common R-structure and just yields the abelian group $M \otimes_R N$. It is instructive to see that tensoring functors are not necessarily left exact, though. In other words, they do not need to preserve kernels. Even in the special case of abelian groups $R = \mathbb{Z}$ with $\otimes_{\mathbb{Z}} = \otimes$ we can easily come up with an example that demonstrates what can go wrong.
The canonical inclusion $\chi \colon 2\,\mathbb{Z} \hookrightarrow \mathbb{Z}$ of the even integers into all integers clearly is an injective group homomorphism. However, tensoring with the abelian group $M = \mathbb{Z}/2\,\mathbb{Z}$ destroys injectivity since

$$\begin{aligned} \mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}} \otimes \chi \colon (\mathbb{Z}/2\,\mathbb{Z}) \otimes (2\,\mathbb{Z}) &\to (\mathbb{Z}/2\,\mathbb{Z}) \otimes \mathbb{Z} \\ [1] \otimes 2 \neq 0 &\mapsto [1] \otimes 2 \cdot 1 = [1] \cdot 2 \otimes 1 = [0] \otimes 1 = 0. \end{aligned}$$

So, roughly speaking, if we have a subgroup $G \subseteq H$, we cannot even assume $M \otimes G \subseteq M \otimes H$ in general since the "same" element $m \otimes g \neq 0 \in M \otimes G$ might become zero viewed in $M \otimes H$. In our case, the problem is that in the target group of χ we suddenly have access to the odd integer $1 \in \mathbb{Z} \backslash 2\,\mathbb{Z}$. It allows to shift the factor 2 to the other side of $\otimes$, where it annihilates the finite group of order 2 completely. This example shows for abelian groups that there are new elements $m \otimes g$ in the kernel of $\mathrm{Id}_M \otimes \chi$ for an injective map $\chi\colon G \to H$ whenever it is allowed to divide the embedded $\chi(g) \in H$ by the order of m:

$$\exists\, n \in \mathbb{Z},\, h \in H\colon n \cdot m = 0,\, \chi(g) = n \cdot h \implies m \otimes g \in \ker(\mathrm{Id}_M \otimes \chi).$$

Studying the origin of our counterexample already indicates that torsion conditions play a central role if one aims to measure the failure of left exactness for those tensoring functors by algebraic means. In fact, this analysis leads to the notion of torsion groups $\mathrm{Tor}(M, G)$ for any abelian group G, which turn out to be functorial in G. Given a short exact sequence of abelian groups

$$0 \to G \to H \to I \to 0$$

this functor $\mathrm{Tor}(M, -)$ also comes with a canonical connection map $\mathrm{Tor}(M, I) \to M \otimes G$. This allows to extend the image sequence under $M \otimes (-)$ further to the left and even provides a long exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}(M,G) \arrow[r] & \mathrm{Tor}(M,H) \arrow[r] & \mathrm{Tor}(M,I) \arrow[dll] & \\ & M \otimes G \arrow[r] & M \otimes H \arrow[r] & M \otimes I \arrow[r] & 0. \end{tikzcd}$$

So, in some sense, $\mathrm{Tor}(M, -)$ "repairs" the lack of left exactness for $M \otimes (-)$ on the level of abelian groups. For general module theory apart from $R = \mathbb{Z}$, however, left exactness of the corresponding Tor-functor is no longer guaranteed and the idea is to iterate this prolongation theme on the left. Indeed, in homological algebra there is the concept of left derived functors $(L^n F)_{n \in \mathbb{N}}$ for a right exact functor $F = L^0 F$ between abelian categories, like module categories, for example. They allow to extend the image of a short exact sequence

$$F(M) \to F(N) \to F(P) \to 0$$

under F on the left to obtain a long exact sequence in the end. For this, there are connection maps $(L^n F(P) \to L^{n-1} F(M))_{n \in \mathbb{N}}$ that connect the image sequences under the respective functors iteratively to a long exact sequence

$$\begin{tikzcd}
 & \vdots & \\
L^nF(M) \arrow[r] & L^nF(N) \arrow[r] & L^nF(P) \arrow[dll] \\
L^{n-1}F(M) \arrow[r] & L^{n-1}F(N) \arrow[r] & L^{n-1}F(P) \\
 & \vdots & \\
F(M) \arrow[r] & F(N) \arrow[r] & F(P) \arrow[r] & 0.
\end{tikzcd}$$

To learn more about derived functors and Tor as a special case for tensoring functors, see also [19].

Remark 5.3 ([18, p. 23]). The same idea of derived functors also arises in a somewhat dual flavour for left exact functors. They should allow to extend the short left exact image sequence further to the right. In this context, the representable Hom-functors from Proposition 2.14 are of special interest as the most famous left exact functors in some sense.

Both the Hochschild homology and cohomology materialise as the derived functors for $M \otimes_{A,A} (-)$ and $\mathrm{Hom}_{A,A}(-, M)$, respectively. By this means, they measure invariants of projective approximations for an A-bimodule M.

Definition 5.4 ([20, pp. 8-9]). Let A be a unital algebra and let M be a unital A-bimodule. Define $C_n(A, M) := M \otimes A^{\otimes n}$ for $n \in \mathbb{N}$ and $C_0(A, M) := M$. As in Definition 4.8, we introduce *face maps* $d_i \colon C_n(A, M) \to C_{n-1}(A, M)$ for $0 \leq i \leq n$ as the linear maps induced by

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_n) := \begin{cases} ma_1 \otimes \cdots \otimes a_n, & i = 0, \\ m \otimes a_1 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n, & 1 \leq i < n, \\ a_nm \otimes a_1 \otimes \cdots \otimes a_{n-1}, & i = n. \end{cases}$$

The alternating sum over the face maps

$$b = b_n := \sum_{i=0}^{n} (-1)^i d_i \colon C_n(A, M) \to C_{n-1}(A, M)$$

is called the *Hochschild boundary map.* It turns $(C_\bullet(A, M), b)$ into the *Hochschild chain complex.* We define

$$H_n(A, M) := \frac{\ker(b_n)}{\operatorname{Im}(b_{n+1})}$$

as the n-th *Hochschild homology with coefficients in* M. It consists of n-cycles modulo n-boundaries and also makes sense for $n = 0$ by formally setting $b_0 := 0$. If we take $M = A$ itself, then we write

$$HH_n(A) := H_n(A, A).$$

Lemma 5.5 ([20, p. 4])**.** *The above chain complex is well-defined, that is, $b \circ b = 0$.*

Proof. We can treat the face maps exactly as in Definition 4.8 if we count M as the zeroth factor. For this, we think of the entries building a circle with an imaginary n-th separator that prevents the n-th entry to get multiplied with the 0-th entry. This allows to interpret that the face map d_i removes the i-th separator for all $0 \leq i \leq n$. Now, the computation of $b \circ b = 0$ works as in Definition 4.8. □

Example 5.6. Spelled out for a unital algebra A and a unital bimodule M, we have

$$\begin{aligned} b_1 \colon M \otimes A &\to M \\ m \otimes a &\mapsto ma - am. \end{aligned}$$

So the zeroth Hochschild homology group is the commutator quotient as in Lemma 2.6:

$$H_0(A, M) = M_\# = M/_{[A, M]}.$$

This example already shows that the Hochschild homology provides a non-trivial theory. It does not always vanish such as the homology of the bar complex in Definition 4.8 does. The main difference between the boundary maps b and b' is that the Hochschild boundary also makes use of the ultimate face map that relates the outer entries. In fact, the Hochschild complex is the image chain complex of $\mathrm{Bar}_\bullet$ under the two-sided balanced tensoring functor $M \otimes_{A,A} (-)$, just as the Hochschild cocomplex is the image under the bimodule Hom-functor in Proposition 4.9.

Lemma 5.7 ([20, p. 13]). *Let A be a unital algebra and let M be a non-degenerate bimodule. Then the image of the bar complex under $M \otimes_{A,A} (-)$ yields the Hochschild complex*

$$(M \otimes_{A,A} \mathrm{Bar}_\bullet, \mathrm{Id}_M \otimes_{A,A} b') \cong (C_\bullet(A, M), b).$$

Proof. The linear map defined by

$$\begin{aligned} \kappa \colon M \otimes A^{\otimes n} &\to M \otimes_{A,A} A^{\otimes(n+2)}, \\ m \otimes x &\mapsto m \otimes_{A,A} (1_A \otimes x \otimes 1_A), \end{aligned}$$

is injective since M is assumed to be non-degenerate. It is also surjective because for any $a_0, a_{n+1} \in A$ we may transfer the respective factors on pure tensors to obtain the desired shape

$$m \otimes_{A,A} a_0 \cdot (1_A \otimes x \otimes 1_A) \cdot a_{n+1} = a_{n+1} m a_0 \otimes_{A,A} (1_A \otimes x \otimes 1_A).$$

This isomorphism of vector spaces allows to translate the action of the boundary map $\mathrm{Id}_M \otimes_{A,A} b'_n$ to a boundary map

$$\begin{aligned} \tilde{b}_n \colon M \otimes A^{\otimes n} &\to M \otimes A^{\otimes(n-1)} \\ m \otimes a_1 \otimes \cdots \otimes a_n &\mapsto \sum_{j=0}^{n} (-1)^j \kappa^{-1}(m \otimes_{A,A} d_j(1_A \otimes a_1 \otimes \cdots \otimes a_n \otimes 1_A)). \end{aligned}$$

In fact, only in the first and in the last summand we cannot read off κ^{-1} immediately and we compute

$$\kappa^{-1}(m \otimes_{A,A} d_j(1_A \otimes a_1 \otimes \cdots \otimes a_n \otimes 1_A)) = \begin{cases} m a_1 \otimes \cdots \otimes a_n, & j = 0, \\ d_j(m \otimes a_1 \otimes \cdots \otimes a_n), & 1 \leq j < n, \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}, & j = n. \end{cases}$$

A comparison with Definition 5.4 shows that $\tilde{b} = b$ is indeed the Hochschild boundary, as claimed. □

At this point, we want to import two results from homological algebra to benefit from this interpretation in terms of the bar complex.

Proposition 5.8 ([24, Cor. 17.15], [18, III, Prop. 6.1]). *Let A be a unital algebra and let N be a unital bimodule. Then all projective A-bimodule resolutions of M are homotopy* *equivalent as chain complexes of A-bimodules.*

Proposition 5.9 ([24, Prop. 17.13], [18, V, Thm. 8.1]). *Let A be a unital algebra and let $\varphi_\bullet\colon C_\bullet \to D_\bullet$ be a homotopy equivalence between two complexes of unital A-bimodules. Let F be an additive functor on the category of A-bimodules. Then $F(\varphi_\bullet)$ induces an isomorphism on homology*

$$H_n(F(C_\bullet)) \cong H_n(F(D_\bullet)).$$

Theorem 5.10 ([18, V, Thm. 8.1]). *Let A be a unital algebra and let M be a unital bimodule. Let $P_\bullet \to A$ be a projective bimodule resolution of A. Then for $n \in \mathbb{N}_0$ we have*

$$H_n(A, M) \cong H_n(M \otimes_{A,A} P_\bullet).$$

Proof. The bar resolution in Definition 4.8 provides a projective bimodule resolution for A. So Lemma 5.7 shows the claim for the special case of $\mathrm{Bar}_\bullet$. Hence, it is left to show that the result is independent of the choice of $P_\bullet$. For this, we can make use of Proposition 5.8 for $N = A$. It says that any two projective bimodule resolutions $P_\bullet$ and $P'_\bullet$ are homotopy equivalent by some homotopy equivalence $\varphi_\bullet$. If we now apply Proposition 5.9 with the additive tensoring functor $M \otimes_{A,A} (-)$ in the role of F, it follows that both image complexes of $P_\bullet$ and $P'_\bullet$ have isomorphic homology. This shows the claim. □

This theorem highlights why it is so desirable to have a short projective bimodule resolution $P_\bullet$ for our algebra of interest. The crucial observation is that it can be used to compute the Hochschild homology since it is allowed to replace the standard bar resolution by $P_\bullet$. However, a relative Leavitt path algebra is not necessarily unital. To also treat this case, we further need to develop the theory of Hochschild homology for general algebras.

Example 5.11 ([20, p. 10]). The Hochschild complex for the ground field $A = M = k$ itself takes an easy shape because for any $n \in \mathbb{N}$ the multiplication maps allow to identify $k^{\otimes n} \cong k$. Thus, $b_n = \sum_{i=0}^{n} (-1)^i \mathrm{Id}_k$ alternates between an automorphism for even and the zero homomorphism for odd indices $n \in \mathbb{N}$. In any case, we have ${\{0\}}/{\{0\}} = 0$ and ${k}/{k} = 0$. As a result,

$$HH_n(k) = \begin{cases} {k}/{[k,k]} = k, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Lemma 5.12. *Taking the Hochschild homology $HH_n(-)$ is functorial. That is, any algebra homomorphism $f\colon A \to A'$ between unital k-algebras A and A' induces a homomorphism*

$$\begin{aligned} f_*\colon HH_n(A) &\to HH_n(A'), \\ a_0 \otimes \cdots \otimes a_n &\mapsto f(a_0) \otimes \cdots \otimes f(a_n) \end{aligned}$$

on homology that defines the functor $HH_n(-)$ on the level of morphisms.

Proof. First look at the linear map defined by

$$\begin{aligned} F\colon C_n(A, A) &\to C_n(A', A'), \\ a_0 \otimes \cdots \otimes a_n &\mapsto f(a_0) \otimes \cdots \otimes f(a_n). \end{aligned}$$

We claim that it commutes with the respective Hochschild boundaries. Indeed, since f is multiplicative, we have $d_i \circ F = F \circ d_i$ for any face map with $0 \leq i \leq n$. So $b^{A'} \circ F = F \circ b^A$ follows by linearity. Therefore, F maps cycles to cycles and boundaries to boundaries such that the claimed homomorphism $f_*\colon HH_n(A) \to HH_n(A')$ is well-defined. The properties $(\mathrm{Id}_A)_* = \mathrm{Id}_{HH_n(A)}$ and $(fg)_* = f_* g_*$ for composable algebra homomorphisms follow immediately. □

The functors $(HH_n)_{n\in\mathbb{N}}$ turn out to be additive. Even more, they serve as left derived functors for $HH_0(-) = (-)_\#$, which takes the commutator quotient. But, so far, we only considered unital algebras. Since HH_n is additive, we may apply a standard procedure to extend it from unital to all algebras. For this, we pass to the unitalisation and drop the extra information for the unit summand k afterwards.

Definition 5.13 ([20, p. 28])**.** Let A be a non-unital algebra. Consider the inclusion homomorphism $k \to A^+$ of k into the unitalisation A^+. Then its Hochschild homology is defined as

$$HH_n(A) := \mathrm{coker}(HH_n(k) \to HH_n(A^+)).$$

Lemma 5.14. *Let A be an algebra. Then we have*

$$HH_n(A) = \begin{cases} A/[A, A] = A_\#, & n = 0, \\ HH_n(A^+), & n \neq 0. \end{cases}$$

Proof. By Example 5.11, the Hochschild homology for k is trivial for $n \neq 0$ and k for $n = 0$. Hence, by the construction in Lemma 5.12, this also holds for the image of the map

$HH_n(k) \to HH_n(A^+)$ in Definition 5.13. So, in practise, the Hochschild homology agrees with the one for its unitalisation, except for a unit summand k that is divided out in the zeroth degree $HH_0(A^+) = A^+_\#$ that is discussed in Example 5.6. But scalar multiples of the unit $1 \in A^+$ commute with any element and we have $[A^+, A^+] = [A, A]$. Hence, they are unaffected and the commutator quotient is of the shape $A^+_\# = {}^{A}\!/_{[A,\,A]} \oplus k$. This finishes the proof. $\square$

Remark 5.15 ([20, pp. 28-32], [2, p. 251]). In principle, the Hochschild complex $(C_\bullet, b)$ in Definition 5.4 is also well-defined for a not necessarily unital algebra A. This allows to define a "naive" Hochschild homology

$$HH_n^{\text{naive}}(A) := \frac{\ker(b_n \colon C_n(A,A) \to C_{n-1}(A,A))}{\operatorname{Im}(b_{n+1} \colon C_{n+1}(A,A) \to C_n(A,A))}.$$

However, HH_n^{naive} in general does not match Definition 5.13 above. Roughly speaking, the reason for this is that this homology is conceptually perturbed by the homology of the bar complex for A. Hence, the naive version does not correctly extend the functors $(HH_n)_{n\in\mathbb{N}}$ beyond unital algebras.
Nevertheless, the naive construction above still gives the correct homology theory as long as the algebra A is *H-unital*. In some sense, this is a weaker form of unitality that still allows to show that the bar complex $\mathrm{Bar}_\bullet$ has a vanishing homology such that the above mentioned perturbation is not visible. This happens if A has local units, for example. In this case, the construction of the contracting homotopy in Lemma 4.8 generalises and can be based on tensoring with a local unit u for all simultaneously involved algebra elements, instead. See also [20].

For our purposes, this subtlety only becomes relevant in degree 1 since we deal with quasi-free algebras that also have local units. In this special case, we can sort out the problem indicated in Remark 5.15 by hand.

Proposition 5.16. *Let A be an algebra with local units. Then there is an isomorphism of commutator quotients*

$$\begin{aligned} \varphi_A \colon (A \otimes A)_\# &\to A, \\ x \otimes y + [A, A \otimes A] &\mapsto -yx. \end{aligned}$$

Furthermore, both bimodule versions of noncommutative forms admit the same commutator *quotients:*

$$\overline{\Omega^1}(A)_\# = \overline{\Omega^1}(A)\big/[A, \overline{\Omega^1}(A)] \cong \Omega^1(A)\big/[A^+, \Omega^1(A)] = \Omega^1(A)_\#.$$

Under these two identifications, the map $\iota_\#$ induced by the inclusion of the reduced bimodule of non-commutative forms $\iota\colon \overline{\Omega^1}(A) \to A \otimes A$ takes the following shape:

$$\begin{aligned} b\colon \big(\Omega^1(A)_\# \cong \overline{\Omega^1}(A)_\#\big) &\to \big(A \cong (A \otimes A)_\#\big), \\ xd(y) + [A, \Omega^1(A)] &\mapsto [x, y]. \end{aligned}$$

Proof. For the first isomorphism, we can proceed as in Lemma 5.7, but it is also quickly verified directly. In fact, this way we better see the analogy to Proposition 4.3 and the inspection of the multiplication map mult. The linear map

$$\begin{aligned} f\colon A \otimes A &\to A, \\ x \otimes y &\mapsto -yx, \end{aligned}$$

is clearly surjective since we have $y = yu = f(-u \otimes y)$ for any $y \in A$ with local unit u. Hence, it is left to show that its kernel is given by $[A, A \otimes A]$. On the one hand, for $a, x, y \in A$, we indeed have

$$f([a, x \otimes y]) = -y(ax) + (ya)x = 0.$$

On the other hand, for any $\sum_{j \le n} x_j \otimes y_j \in \ker(f)$, we can choose a common local unit u for the y_j and obtain

$$\begin{aligned} \sum_{j \le n} x_j \otimes y_j &= \sum_{j \le n} (x_j \otimes u)y_j - 0 \otimes u \\ &= \sum_{j \le n} (x_j \otimes u)y_j - \sum_{j \le n} y_j x_j \otimes u \\ &= -\sum_{j \le n} [y_j, x_j \otimes u] \in [A, A \otimes A]. \end{aligned}$$

This establishes $\ker(f) = [A, A \otimes A]$ and therefore also the first isomorphism. For the second one, we consider the composition of the inclusion $\overline{\Omega^1}(A) \subseteq \Omega^1(A)$ with the quotient map into $\Omega^1(A)_\#$:

$$\psi\colon \overline{\Omega^1}(A) \to \Omega^1(A)_\#.$$

We aim to show that this map is surjective. By Lemma 4.6 it is enough to check that the class of $d(a)$ for $a \in A$ is in the image. Indeed, if u is a local unit for a, then

$$\begin{aligned} d(a) &= ud(a)u + d(u)au + uad(u) \\ &= ud(a)u + ud(u)a + [d(u)a, u] + ad(u)u + [u, ad(u)] \in \overline{\Omega^1}(A) + [A, \Omega^1(A)]. \end{aligned}$$

Now, again by Lemma 4.6, the kernel of ψ is given by

$$\big(A\Omega^1(A)A\big) \cap [A^+, \Omega^1(A)] = [A^+, A\Omega^1(A)A] = [A, \overline{\Omega^1}(A)].$$

This establishes the second isomorphism. The final version of b now follows immediately from $\varphi_{A^+}(x \otimes y - xy \otimes 1) = -yx - (-1 \cdot xy) = xy - yx = [x, y]$. $\square$

Theorem 5.17. *Let L be a relative Leavitt path algebra. Define the linear map*

$$\begin{aligned} b\colon \Omega^1(L)_\# &\to L, &(19)\\ xd(y) + [L, \Omega^1(L)] &\mapsto [x, y]. &(20) \end{aligned}$$

Then the Hochschild homology of L is given by the cokernel and the kernel of b:

$$HH_n(L) = \begin{cases} \operatorname{coker}(b) = L_\#, & n = 0, \\ \ker(b), & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Proof. Let A be a unital algebra. By Theorem 5.10 we know that its Hochschild homology is given by the homology of the image chain complex under $A \otimes_{A,A} (-)$ for any projective bimodule resolution of A. Since L is quasi-free by Theorem 3.18, we may always take the resolution (13) for $A = L^+$. Now, Proposition 5.1 applies and gives that the Hochschild homology of L^+ is nothing but the homology of

$$\ldots \xrightarrow{0} 0 \xrightarrow{0} C_1 = (\Omega^1(L))_\# \xrightarrow{\iota_\#} C_0 = (L^+ \otimes L^+)_\#.$$

Since $\iota_\#$ is the only non-zero boundary map, we have that the Hochschild homology of L^+ vanishes for $n \geq 2$. Furthermore, it is given by the cokernel of $\iota_\#$ in the zeroth degree and by its kernel in the first degree. Now we are in a position to apply Proposition 5.16 for

$A = L^+$. It says that we may equivalently consider the map

$$\begin{aligned} b_+ \colon \Omega^1(L)_\# &\to L^+ \\ xd(y) + [L, \Omega^1(L)] &\mapsto [x, y]. \end{aligned}$$

By Lemma 5.14 we need to divide out a further unit summand k in zeroth degree to read off the Hochschild homology for L itself. Hence, we arrive at the cokernel and the kernel of the map

$$\begin{aligned} b \colon \Omega^1(L)_\# &\to L \\ xd(y) + [L, \Omega^1(L)] &\mapsto [x, y]. \end{aligned}$$

This shows the claim. □

Remark 5.18. Note that the detour over L^+ is indeed unnecessary as indicated in Remark 5.15 due to the unproven fact that algebras with local units are H-unital. We could have arrived at the same result immediately if we knew beforehand that it is allowed to use the projective resolution in Theorem 4.22 right away although L itself is nonunital.

Remark 5.19. To see the similarities between b in (19) and $b_1 = d_0 - d_1$ in Definition 5.4, the face map interpretation from Lemma 5.5 turns out to be helpful again. Let $xd(y) \equiv d(y)x \in \Omega^1(L)_\#$. In this context, $d(.)$ takes the role of the separators that prevent x to get multiplied with y and vice versa. Indeed, b is obtained by removing them in either of the two representations $xd(y)$ or $d(y)x$ with alternating sign:

$$b(xd(y)) = xy - yx.$$

To continue with the computation of the Hochschild homology, we should find an explicit form for $\Omega^1(L)_\#$. In fact, Proposition 5.16 shows that it does not matter whether we deal with $\Omega^1(L)_\#$ or $\overline{\Omega^1}(L)_\#$. Hence, the direct sum decomposition in Theorem 4.22 can be used right away. But first it is recommendable to rephrase the second big summand involving $d(p_e) = d(ee^*)$.

Proposition 5.20 ([22]). *Let L be a relative Leavitt path algebra. Then*

$$\begin{aligned}\overline{\Omega^1}(L) = &\bigoplus_{v\neq w\in E^0} Lvd(v)wL \oplus \bigoplus_{e\in s^{-1}(X)} Ls(e)d(e)r(e)L\\ &\oplus \bigoplus_{e\in s^{-1}(E^0\setminus X)} L(p_e+q_e)d(e)r(e)L \oplus \bigoplus_{e\in s^{-1}(E^0\setminus X)} Lr(e)d(e^*)q_eL.\end{aligned}$$

Proof. Use the Leibniz rule and $p_e \perp q_e$ to rephrase the generators in the second big summand in Theorem 4.22 as

$$p_ed(p_e)q_e = p_eed(e^*)q_e + p_ed(e)e^*q_e = ed(e^*)q_e + p_ed(e)e^*(p_eq_e) = ed(e^*)q_e$$

and likewise

$$q_ed(p_e)p_e = q_eed(e^*)p_e + q_ed(e)e^*p_e = (q_ep_e)ed(e^*)q_e + q_ed(e)e^* = q_ed(e)e^*.$$

Furthermore, the $(CK1)$ relation $r(e) = e^*e$ implies equality in $Le = Ler(e) \subseteq Lr(e)$ and $e^*L = r(e)e^*L \subseteq r(e)L$. Hence, we can replace the generators in the former second big summand by $r(e)d(e^*)q_e$ and $q_ed(e)r(e)$. This shows the claim. □

Corollary 5.21. *Let L be a relative Leavitt path algebra. For notational convenience, we introduce the following big direct summands S_j for $0 \leq j \leq 3$:*

$$\begin{aligned}S_0 &:= \bigoplus_{v\neq w\in E^0} wLvd(v),\\ S_1 &:= \bigoplus_{e\in s^{-1}(X)} r(e)Ls(e)d(e),\\ S_2 &:= \bigoplus_{e\in s^{-1}(E^0\setminus X)} r(e)L(p_e+q_e)d(e),\\ S_3 &:= \bigoplus_{e\in s^{-1}(E^0\setminus X)} q_eLr(e)d(e^*).\end{aligned}$$

Then we have $\Omega^1(L)_\# \cong S_0 \oplus S_1 \oplus S_2 \oplus S_3$.

Proof. By Proposition 5.16 we may take the commutator quotient for the reduced bimodule of noncommutative forms as well. Now, in every summand of Proposition 5.20 we can use commutators to move the algebra elements on the right of $d(.)$ to the left. This gives a unique representation of the claimed form. □

Remark 5.22. The representation in Corollary 5.21 gives a handy direct sum decomposition for the domain of b in (19). It tells us, which representatives of differential forms are the relevant ones for each of the generators $x \in E^0 \cup E^1 \cup s^{-1}(E^0 \setminus X)^*$ that represent the unlocalised case in some sense. Their associated commutators suffice to span all other commutators because of $\mathrm{Im}(b) = [L, L]$.
Further note that $HH^1(L) = \ker(b)$ and $L_\# = \mathrm{coker}(b)$ in Theorem 5.17 only measure the failure of b being injective or surjective, respectively. Roughly speaking, we can neglect bijective parts of b without changing them. Concretely, for any subspace $S \subseteq \Omega^1(L)_\#$, on which b acts injectively, we may pass to the map on the quotients

$$\bar{b}\colon \Omega^1(L)_\#/S \to L/b(S)$$

without changing the kernel or cokernel. In fact, this observation highlights our further strategy of dividing out exact subcomplexes and first applies for S_0.

Lemma 5.23. *Let the summands S_j be as in Corollary 5.21. Then the Hochschild homology of L is given by the kernel and the cokernel of the linear map*

$$\begin{aligned} S_1 \oplus S_2 \oplus S_3 &\to \bigoplus_{v \in E^0} vLv \\ xd(y) &\mapsto [x, y]. \end{aligned}$$

Proof. By Theorem 5.17 and 5.21 we have that the Hochschild homology is given by the kernel and the cokernel of the linear map

$$\begin{aligned} b\colon S_0 \oplus S_1 \oplus S_2 \oplus S_3 &\to L \\ xd(y) &\mapsto [x, y]. \end{aligned}$$

However, b acts injectively on the summand

$$S_0 = \bigoplus_{v \neq w \in E^0} wLvd(v)$$

since for any $a \in L$ and any vertex pair $v \perp w$ we have that

$$b(wavd(v)) = [wav, v] = wav^2 - vwav = wav.$$

So $b|_{S_0}$ is bijective onto its image $\bigoplus_{v \neq w} wLv$ with the obvious inverse $wav \mapsto wavd(v)$. Hence, the kernel and the cokernel stay the same if we divide out S_0 in the domain and its image $b(S_0)$ in the range. Since this amounts to leaving out the corresponding direct summands, the resulting linear map is of the claimed shape:

$$S_1 \oplus S_2 \oplus S_3 = {}^{\Omega^1(L)_\#}\!/_{S_0} \to \bigoplus_{v \in E^0} vLv = {}^{L}\!/_{b(S_0)}. \qquad \square$$

To construct more complicated linear maps in the inverse direction and to keep track of further computations with b, it is of advantage to have a basis for L at hand.

Definition 5.24 ([2, Def. 1.5.11])**.** Let E be a quiver and let $\mathcal{B}$ be the basis of the Cohn algebra $C(E)$ from Proposition 3.7. Let $X \subseteq \mathrm{Reg}(E)$ and let $L := L_k^X(E)$ be the corresponding relative Leavitt path algebra viewed as quotient of $C(E)$. Enumerate the fibre $s^{-1}(x)$ for a given $x \in X$ and call its maximal element the *maximal edge* e_x. In this context, we define

$$\mathcal{B}_X := \mathcal{B} \setminus \{\mu e_x e_x^* \nu^* \mid \mu, \nu \in \mathrm{Path}(E),\, r(\mu) = r(\nu) = x \in X\}$$

as the *standard basis* of L. For $m \in \mathbb{Z}$, its subset

$$\mathcal{B}_{X,m} := \{\mu\nu^* \in \mathcal{B}_X \colon |\mu| - |\nu| = m\}$$

is called the *m-basis* of L.

Lemma 5.25. *The standard basis in Definition 5.24 is indeed a k-basis of L. If $L = \bigoplus_{m \in \mathbb{Z}} L_m$ denotes the $\mathbb{Z}$-grading in Corollary 3.8, then the m-basis provides a basis of L_m and can be rewritten as*

$$\mathcal{B}_{X,m} = \begin{cases} \{\alpha\theta\beta^* \in \mathcal{B}_X \colon |\alpha| = |\beta|,\, \theta \in E^m\}, & m \geq 0, \\ (\mathcal{B}_{X,-m})^*, & m < 0. \end{cases}$$

Proof. The proof is based on Proposition 3.7 and Remark 4.19. The ideal generated by the residual source projections in the Cohn algebra

$$C(E)\{q_{e_x} \mid x \in X\}C(E)$$

can be further simplified. By Proposition 4.18 and 3.14 we have $q_{e_x} \perp (ff^*)$ for all edges

$f \in E^1$. Thus, $q_{e_x} f = 0 = f^* q_{e_x}$ and the ideal is already spanned by compatible words of the shape

$$\mathcal{B}' := \left\{ \mu q_{e_x} \nu^* = \mu\nu^* - \sum_{e \in s^{-1}(x)} \mu e e^* \nu^* \,\middle|\, r(\mu) = r(\nu) = x \in X \right\}.$$

They are also linearly independent over the Cohn algebra due to Proposition 3.7. Hence, we found a basis for the ideal and it suffices to show that $\mathcal{B}' \cup \mathcal{B}_X$ provides a basis for the Cohn algebra $C(E)$. Indeed, any missing basis word in $\mathcal{B}$ can be written as

$$\mu e_x e_x^* \nu^* = \mu\nu^* - \mu q_{e_x} \nu^* - \sum_{e < e_x} \mu e e^* \nu^*.$$

Furthermore, any non-zero linear combination of words in $\mathcal{B}'$ admits a summand $\mu e_x e_x^* \nu^*$ with non-zero coefficient by definition of q_{e_x}. Hence, it cannot lie in $\mathcal{B}_X$ and this finally shows that the quotient algebra L has the desired basis $\mathcal{B}_X = (\mathcal{B}' \cup \mathcal{B}_X) \setminus \mathcal{B}'$.
Note that all basis words in $\mathcal{B}_X$ are homogeneous elements with respect to the $\mathbb{Z}$-grading in Corollary 3.8. We can sort them by their degrees and obtain that $\mathcal{B}_{X,m}$ is a basis of L_m. For the final claim, let $\mu\nu^* \in \mathcal{B}_{X,m}$ be any basis word. Depending on the sign of m, either μ or ν is longer by exactly $|m|$ edges. This extra portion θ can be stated explicitly in our basis word. For $m \geq 0$ we relabel the paths $\mu = \alpha\theta$ and $\nu = \beta$ to obtain the desired shape $\alpha\theta\beta^*$. Likewise for $m < 0$ with $\mu = \alpha$ and $\nu = \beta\theta$, which results in $\alpha\theta^*\beta^*$. Alternatively, the *-involution flips the sign of the degree and allows to reduce to the first case directly. □

Remark 5.26. For any $m \geq 0$, the notation for the basis word $\alpha\theta\beta^* \in \mathcal{B}_{X,m}$ is designed to provide access to the outer paths α and β of the same length, say n. Recall from the reduction step in Proposition 3.7 that for paths of the same length the n-fold application of the $(CK1)$ relation yields $\beta^*\alpha = \delta_{\beta,\alpha} r(\beta)$. Thus,

$$\begin{aligned}
[\alpha\theta, \beta^*] &= \alpha\theta\beta^* - \beta^*\alpha\theta \\
&= \alpha\theta\beta^* - \delta_{\beta,\alpha}\,\theta \\
&= \alpha\theta\beta^* - \theta\beta^*\alpha \\
&= [\alpha, \theta\beta^*].
\end{aligned}$$

This observation shows that not only elements with different outer vertices but also the lately established basis words with different outer paths can be written as commutators.

Furthermore, if the outer paths agree, then $\alpha\theta\alpha^*$ and θ are the same modulo commutators. So, with Lemma 5.23 in mind, we can already limit ourselves to paths θ with $s(\theta) = r(\theta)$ and their corresponding ghost paths θ^* to span the commutator quotient $L_\#$.

Definition 5.27. Let E be a quiver. Define the set of *closed paths* as

$$\mathrm{Path}_c(E) := \{\theta \in \mathrm{Path}(E) \mid s(\theta) = r(\theta)\}.$$

For any closed path $\theta \in \mathrm{Path}_c(E)$, we say that it is *based at* $v = s(\theta) = r(\theta)$. Since any closed path has a unique length and a unique base point, the set of closed paths is the disjoint union of the sets

$$\mathcal{C}_{v,m} := \{\theta \in E^m \mid s(\theta) = r(\theta) = v\}, \quad v \in E^0, \quad m \in \mathbb{N}_0\,.$$

If there are only the vertices as closed paths, that is, $\mathcal{C}_{v,m} = \emptyset$ for $m \neq 0$, then the quiver is called *acyclic*.
A closed path of length $m = 1$ is called a *loop*. For the extended quiver, this notion can be consistently extended to $m < 0$ by the set of *closed ghost paths*

$$\mathcal{C}_{v,m} := \mathcal{C}^*_{v,-m}.$$

The next goal is to refine the idea of Remark 5.26 in terms of the commutators that are highlighted by the generators of the summands S_j. Ultimately, we want to arrive at a linear map $\psi\colon L \to \Omega^1(L)_\#$ in the inverse direction to b that serves as some kind of "reduction algorithm". That is, it should allow to reduce basis words in L modulo commutators via powers of $\mathrm{Id}_L - b\psi$. In this way, we aim to limit our search for suitable representatives in $L_\#$ to closed paths systematically.
The main idea for the definition of ψ is to separate an outer vertex or edge from a reducible basis word. Roughly speaking, it gets stored inside the $d(.)$-part in $\Omega^1(L)_\#$. With the separator interpretation from Remark 5.19 in mind, $b\psi$ now returns the word we started with and an exchange term. This exchange term has the selected vertex or edge on the other end of the word and therefore simplifies by using the vertex orthogonality or the $(CK1)$ relation, respectively.

Definition 5.28 ([22])**.** Let $\psi\colon L \to \Omega^1(L)_\#$ be the linear map that is defined by the following assignment for basis words $\mu\nu^* \in \mathcal{B}_X$ down below. If both paths have length ≥ 1, then the definition uses that there is a leading edge that can be distinguished from the rest of the path, that is, $\mu = \alpha_1\mu_r$ and $\nu = \beta_1\nu_r$:

$$\mu\nu^* \mapsto \begin{cases} \mu\nu^* d(s(\nu)) \in S_0, & s(\mu) \neq s(\nu), \\ 0, & s(\mu) = s(\nu),\ |\mu| = 0 \vee |\nu| = 0, \\ -\mu_r\nu^* d(\alpha_1) \in S_1, & s(\mu) = s(\nu) \in X,\ |\mu| \geq 1,\ |\nu| \geq 1, \\ -\mu_r\nu^* d(\alpha_1) \in S_2, & s(\mu) = s(\nu) \notin X,\ |\mu| \geq 1,\ |\nu| \geq 1,\ \alpha_1 \leq \beta_1, \\ \mu\nu_r^* d(\beta_1^*) \in S_3, & s(\mu) = s(\nu) \notin X,\ |\mu| \geq 1,\ |\nu| \geq 1,\ \alpha_1 > \beta_1. \end{cases}$$

Proposition 5.29 ([22]). *Let ψ be as in Definition 5.28. Then the indicated summand memberships hold and for any basis word $\mu\nu^* \in \mathcal{B}_X$ the sequence*

$$\left((\mathrm{Id}_L - b\psi)^n(\mu\nu^*)\right)_{n\in\mathbb{N}_0} \tag{21}$$

becomes constant after at most $\min(|\mu|, |\nu|)$ iterations. Thus, it becomes constant for any

$$a = \sum_j \lambda_j \mu_j \nu_j^* \in L, \quad \lambda_j \in k, \quad \mu_j\nu_j^* \in \mathcal{B}_X$$

after $n(a) := \max_j(\min(|\mu_j|, |\nu_j|))$ iterations as well. Likewise, for any element $\omega \in \Omega^1(L)_\#$ the sequence

$$\left((\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)^n(\omega)\right)_{n\in\mathbb{N}_0} \tag{22}$$

also becomes constant after a finite number of iterations $n(\omega)$ since it does on basis elements for $\Omega^1(L)_\#$.

Proof. Consider the cases that arise in the definition of ψ above. In the first case, the basis words $\mu\nu^*$ with different outer vertices $s(\mu) \neq s(\nu)$ span the subspace $\bigoplus_{v\neq w} wLv$. Moreover, the restrictions $\psi|_{\bigoplus_{v\neq w} wLv}$ and $b|_{S_0}$ are inverses of each other by Lemma 5.23. So both sequences (21) and (22) vanish after the first iteration.

In the second case, we may as well use the notation in Definition 5.27 and note that ψ is defined to be zero on

$$\bigcup_{v\in E^0} \bigcup_{m\in\mathbb{Z}} \mathcal{C}_{v,m}.$$

For these critical basis words we aim for no reduction by now and $(\mathrm{Id}_L - b\psi)^n$ acts trivially throughout.

In the third case and beyond, we deal with a mixture of paths and ghost paths with common outer vertices. Hence, we can highlight the first edges $\mu = \alpha_1\mu_r$ and $\nu = \beta_1\nu_r$

with $s(\alpha_1) = s(\beta_1)$ as in Definition 5.28. If this vertex is in X, then $r(\alpha_1) = s(\mu_r)$ and $s(\alpha_1) = s(\nu)$ already suffice to see $\psi(\mu\nu^*) \in S_1$. Otherwise for $\alpha_1, \beta_1 \in s^{-1}(E^0 \setminus X)$ we have by definition of the residual source projections in 4.17 that

$$\begin{aligned}
\alpha_1 \leq \beta_1 &\implies \beta_1^*(q_{\alpha_1} + p_{\alpha_1}) = \beta_1^* s(\alpha_1) - \sum_{e<\alpha_1} \beta_1^* p_e = \beta_1^* - \sum_{e<\alpha_1} \beta_1^* p_{\beta_1} p_e = \beta_1^*, \\
\alpha_1 > \beta_1 &\implies q_{\beta_1}\alpha_1 = s(\beta_1)\alpha_1 - \sum_{e\leq\beta_1} p_e\alpha_1 = \alpha_1 - \sum_{e\leq\beta_1} p_e p_{\alpha_1}\alpha_1 = \alpha_1.
\end{aligned}$$

This establishes the claimed summand memberships

$$\psi(\mu\nu^*) \in \begin{cases} r(\alpha_1)Ls(\alpha_1)d(\alpha_1) \subseteq S_1, & s(\alpha_1) \in X, \\ r(\alpha_1)L(q_{\alpha_1} + p_{\alpha_1})d(\alpha_1) \subseteq S_2, & \alpha_1 \leq \beta_1, \\ q_{\beta_1}Lr(\beta_1)d(\beta_1^*) \subseteq S_3, & \alpha_1 > \beta_1. \end{cases}$$

In any of the currently considered cases, $b\psi$ is designed to return the basis word we started with as the part of the commutator with positive sign. In the second part, an outer edge is transferred to the other side and allows to apply the $(CK1)$ relation. Concretely:

$$(\mathrm{Id}_L - b\psi)(\mu\nu^*) = \mu\nu^* - \left(\mu\nu^* - \begin{cases} \mu_r\nu^* \cdot \alpha_1 \\ \beta_1^* \cdot \mu\nu_r^* \end{cases}\right) = \delta_{\alpha_1,\beta_1}\mu_r\nu_r^*.$$

Hence, in case of $|\mu| \geq 1$ and $|\nu| \geq 1$, every iteration of $(\mathrm{Id}_L - b\psi)$ strips off a common pair of outer edges $\alpha_1(.)\beta_1^*$ if they agree or arrives at zero immediately. So, after at most $\min(|\mu|, |\nu|)$ iterations in (21), all treated computations either land at zero or at a critical basis word. That is, at a vertex, at a non-trivial closed path θ or at a non-trivial closed ghost path θ^*.
When it comes to iterations of $(\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)$ in (22), we can investigate the results summandwise. For $q_e Lr(e)d(e^*) \subseteq S_3$, we only need to consider basis elements

$$q_e\mu\nu^* d(e^*) \in S_3 \tag{23}$$

with $\mu\nu^* \in \mathcal{B}_X$, $s(\mu) = s(e) \notin X$ and $s(\nu) = r(e)$. If $\mu = \alpha_1\mu_r$ is non-trivial, then we may further assume $\alpha_1 > e$ because otherwise the expression in (23) vanishes. Moreover, if we apply b to (23), the exchange term vanishes because we have $e^* q_e = 0$. So for the effect of ψb, it is left to compute $\psi(q_e\mu\nu^* e^*)$. Note that if the path part is non-trivial, that is

$|\mu| \geq 1$, then our assumptions take care that we have $q_e\alpha_1 = \alpha_1$ and also get back the element we started with. For $|\mu| = 0$, however, $q_e\nu^*$ has to be spelled out and only the closed ghost path ν^*e^* itself vanishes under ψ:

$$\psi(q_e\mu\nu^*e^*) = \begin{cases} \psi(\alpha_1\mu_r\nu^*e^*) = \alpha_1\mu_r\nu^*d(e^*) = q_e\mu\nu^*d(e^*), & |\mu| \geq 1, \\ \psi(\nu^*e^*) - \sum_{f\leq e}\psi(ff^*\nu^*e^*) = \sum_{f\leq e} f^*\nu^*e^*d(f), & |\mu| = 0. \end{cases} \tag{24}$$

So already the first iteration of $(\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)$ yields zero on S_3 unless the word in front of $d(e^*)$ has a trivial path part $|\mu| = 0$. But even then $(\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)(q_e\nu^*d(e^*))$ simplifies in a convenient way. In fact, rearranging terms in $\Omega^1(L)_\#$ shows:

$$\begin{aligned} (\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)(q_e\nu^*d(e^*)) &= q_e\nu^*d(e^*) - \sum_{f\leq e} f^*\nu^*e^*d(f) \\ {\scriptstyle[\,\text{Leibniz}\,]} &= q_e\nu^*d(e^*) + \sum_{f\leq e} f^*\nu^*d(e^*)f - \sum_{f\leq e} f^*\nu^*d(e^*f) \\ {\scriptstyle[\,\text{shift}\, f]} &\equiv q_e\nu^*d(e^*) + \sum_{f\leq e} ff^*\nu^*d(e^*) - r(e)e^*\nu^*r(e)d(r(e)) \\ {\scriptstyle[\,\text{shift}\, r(e)]} &\equiv \nu^*d(e^*) - e^*\nu^*r(e)d(r(e))r(e) \\ {\scriptstyle[(15)]} &= \nu^*d(e^*). \end{aligned}$$

Since both ν^*e^* and $e^*\nu^*$ are closed ghost paths by assumption, this element clearly vanishes under ψb. Thus, it remains invariant under further iterations. To sum up, for every $n \in \mathbb{N}$ and any element in (23) we have:

$$(\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)^n(q_e\mu\nu^*d(e^*)) = \begin{cases} 0, & |\mu| \geq 1, \\ \nu^*d(e^*), & |\mu| = 0. \end{cases}$$

This shows the claim for S_3. We turn to the discussion of (22) for both S_1 and S_2 now. If we have a look at $e \in s^{-1}(X)$ and $r(e)Ls(e)d(e) \subseteq S_1$, we only need to consider elements

$$\mu\nu^*d(e) \in S_1 \tag{25}$$

with $\mu\nu^* \in \mathcal{B}_X$, $s(\mu) = r(e)$ and $s(\nu) = s(e) \in X$. Likewise for $e \in s^{-1}(E^0 \setminus X)$ and $r(e)L(p_e + q_e)d(e) \subseteq S_2$, where we look at elements

$$\mu\nu^*(p_e + q_e)d(e) \in S_2 \tag{26}$$

with $\mu\nu^* \in \mathcal{B}_X$, $s(\mu) = r(e)$ and $s(\nu) = s(e) \notin X$. If the ghost path part $\nu = \beta_1\nu_r$ is non-trivial, we may further assume $e \leq \beta_1$ because otherwise the expression in (26) vanishes. Similar to the S_3 discussion above, the computation of ψb now becomes more tractable if we distinguish between $|\nu| = 0$ and $|\nu| \geq 1$.
If the ghost path is trivial, that is $|\nu| = 0$, then we have $b(\mu d(e)) = \mu e - e\mu$ for an element in (25). By assumption this expression consists of two closed paths and therefore vanishes under ψ. For an element in (26), however, the exchange term still admits $(p_e + q_e)$ on the right and we compute:

$$\psi b(\mu(p_e + q_e)d(e)) = \psi(\mu e) - \psi\left(e\mu - \sum_{f<e} e\mu f f^*\right) = \sum_{f<e} \psi(e\mu f f^*) = \sum_{f<e} e\mu f d(f^*).$$

This is not too bad either because in complete analogy to the S_3 discussion we have:

$$\begin{aligned}
(\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)(\mu(p_e + q_e)d(e)) &= \mu(p_e + q_e)d(e) - \sum_{f<e} e\mu f d(f^*) \\
{}_{[\text{shift } e]} &\equiv \mu(p_e + q_e)d(e) - \sum_{f<e} \mu f d(f^*)e \\
{}_{[f^*e]} &= \mu(p_e + q_e)d(e) + \sum_{f<e} \mu f f^* d(e) \\
&= \mu d(e).
\end{aligned}$$

This shows the claim for S_1 and S_2 in case of $|\nu| = 0$.
If we now apply b either to (25) or to (26) with $|\nu| \geq 1$, then we get two words in L. On the one hand, we have $\mu\nu^* e$, which reduces to $\delta_{\beta_1,e}\,\mu\nu_r^*$ with the $(CK1)$ relation. On the other hand, we subtract a second portion $e\mu\nu^*$.
This is the only time during this proof, where we have to be careful whether this is a basis word or not because in general we have

$$\mu\nu^* \in \mathcal{B}_X,\ s(\mu) = r(e),\ s(\nu) = s(e),\ |\nu| \geq 1 \ \not\Rightarrow\ e\mu\nu^* \in \mathcal{B}_X.$$

The additional edge on the left causes trouble if the resulting path and ghost path parts meet at $e_x e_x^*$ for a maximal edge with $x \in X$. This happens if and only if $|\mu| = 0$, $e = e_x$ for some $x \in X$, and $\nu = \theta e_x$ for some closed path $\theta \in \mathcal{C}_{x,m}$ with $m \geq 0$. In this case, we need to decompose this word into a sum of basis words first and obtain:

$$\begin{aligned}(\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)(e_x^*\theta^* d(e_x)) &= e_x^*\theta^* d(e_x) - \delta_{m\geq 1}\delta_{\theta_1, e_x}\, \psi(e_x^*\theta_r^*) + \psi(e_x e_x^* \theta^*) \\ &= e_x^*\theta^* d(e_x) - 0 + \psi(\theta^*) - \sum_{f<e_x} \psi(ff^*\theta^*) \\ &= \sum_{f\leq e_x} f^*\theta^* d(f).\end{aligned}$$

For $m = 0$, that is, $\theta = x$, we have that

$$b\left(\sum_{f\leq e_x} f^* d(f)\right) = \sum_{f\leq e_x} r(f) - x \tag{27}$$

is mapped to 0 under ψ. So the sequence is already constant at this stage. This also holds for $m \geq 1$ with $\theta = \theta_1\theta_r$ since it takes the following form:

$$\begin{aligned}(\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)(e_x^*\theta^* d(e_x)) &= \sum_{f\leq e_x} f^*\theta^* d(f) \\ {\scriptstyle [\text{shift } f^*]}\ &\equiv \sum_{f\leq e_x} \theta^* d(f) f^* \\ {\scriptstyle [\text{Leibniz}]}\ &= \sum_{f\leq e_x} x\theta^* x d(ff^*) - \sum_{f\leq e_x} \theta^* f d(f^*) \\ {\scriptstyle [(CK2)]}\ &= x\theta^* x d(x) - \sum_{f\leq e_x} \theta^* d(f^*) \\ {\scriptstyle [(15)]}\ &= -\sum_{f\leq e_x} \delta_{\theta_1, f}\, \theta_r^* d(f^*) \\ &= -\theta_r^* d(\theta_1^*).\end{aligned}$$

Indeed, both θ^* and $\theta_1^*\theta_r^*$ are closed ghost paths and are mapped to zero under ψ. This shows the claim for those elements in (25) that lead to $e\mu\nu^* \notin \mathcal{B}_X$.

In the remaining cases, we have that $e\mu\nu^* \in \mathcal{B}_X$ is mixed and that its outer edges satisfy either $s(\beta_1) = s(e) \in X$ for the S_1 elements or $s(\beta_1) = s(e) \notin X$ with $e \leq \beta_1$ for the S_2 elements by assumption. Hence, in both cases $\psi(-e\mu\nu^*)$ returns the element we started with, and this shows the following:

$$\begin{aligned}
(\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)(\mu\nu_r^*\beta_1^* d(e)) &= \mu\nu^* d(e) - (\psi(\mu\nu^* e) + \psi(-e\mu\nu^*)) \\
&= \mu\nu^* d(e) - (\delta_{\beta_1,e}\,\psi(\mu\nu_r^*) + \mu\nu^* d(e)) \\
&= -\delta_{\beta_1,e}\,\psi(\mu\nu_r^*) \\
&= \begin{cases} 0, & |\mu| = 0 \vee \nu \neq e\beta_2\nu_{rr}, \\ \mu_r\nu_{rr}^*\beta_2^* d(\alpha_1), & |\mu| \geq 1,\ \nu = e\beta_2\nu_{rr},\ r(e) \in X, \\ \mu_r\nu_{rr}^*\beta_2^* d(\alpha_1), & |\mu| \geq 1,\ \nu = e\beta_2\nu_{rr},\ r(e) \notin X,\ \alpha_1 \leq \beta_2, \\ -\mu\nu_{rr}^* d(\beta_2^*), & |\mu| \geq 1,\ \nu = e\beta_2\nu_{rr},\ r(e) \notin X,\ \alpha_1 > \beta_2. \end{cases}
\end{aligned}$$

In the last case, $|\mu| \geq 1$ is preserved and guarantees that the sequence (22) runs into zero in the next iteration. Otherwise, a non-trivial reduction for a mixed word in front of $d(e)$ strips off its outer edges $\alpha_1(.)e^*$ and puts the remainder in front of $d(\alpha_1)$ instead. Note that in this final case distinction the involved path and ghost path lengths strictly decrease with every non-trivial iteration in (22).
Thus, the sequence (22) terminates after at most $\min(|\nu|, |\mu|) + 1$ iterations. This finally completes the proof. □

Remark 5.30. In the proof of Proposition 5.29 we also computed the limiting words that arise for the sequences

$$((\mathrm{Id}_L - b\psi)^n(\mu\nu^*))_{n\in\mathbb{N}_0} \quad \text{and} \quad \big((\mathrm{Id}_{\Omega^1(L)_\#} - \psi b)^n(\omega)\big)_{n\in\mathbb{N}_0}$$

starting with basis words $\mu\nu^* \in L$ or $\omega \in \Omega^1(L)_\#$.
Except for 0, the limiting words for the first sequence in (21) are precisely the closed paths and ghost paths

$$\bigcup_{m\in\mathbb{Z}} \bigcup_{v\in E^0} \mathcal{C}_{v,m}$$

from Definition 5.27. If $V_{v,m}$ denotes the k-span of $\mathcal{C}_{v,m}$, then the non-trivial limiting words yield the vector space which is the direct sum over all

$$V_m := \bigoplus_{v\in E^0} V_{v,m}, \quad m \in \mathbb{Z}\,.$$

For the other sequence in (22), the limiting words are more diverse, though. Except for 0, we discovered four different limit types. We list them in the order of their appearance in the proof of Proposition 5.29:

- Firstly, $\nu^* d(e^*)$ for any $e \in s^{-1}(E^0 \setminus X)$ and any path ν that starts at $r(e)$ and ends at $s(e)$.
- Secondly, $\mu d(e)$ for any $e \in E^1$ and any path μ that starts at $r(e)$ and ends at $s(e)$.
- Thirdly, $\sum_{f \in s^{-1}(x)} f^* d(f)$ for any $x \in X$. The equation (27) suggests to flip its sign and to work with $\omega_x := -\sum_{f \in s^{-1}(x)} f^* d(f) = \sum_{f \in s^{-1}(x)} f d(f^*)$ instead.
- Lastly, $-\nu^* d(e^*)$ for any $e \in s^{-1}(X)$ and any path ν that starts at $r(e)$ and ends at $s(e)$.

Note that μ and ν may also be vertices if e is a loop itself. Otherwise they are constructed in such a way that the compositions μe and $e\mu$ or $e^*\nu^*$ and $\nu^* e^*$ become closed paths or closed ghost paths, respectively. In total, the non-trivial limiting words yield the vector space

$$\bigoplus_{e \in E^1} \left(r(e)kEs(e)d(e) \oplus s(e)(kE)^* r(e)d(e^*) \right) \oplus \bigoplus_{x \in X} k\omega_x.$$

Theorem 5.31. *Let L be a relative Leavitt path algebra and let $R := \left(\mathrm{Id}_{\Omega^1(L)_\#} - \psi b\right)$ be the reduction operator for the domain of b as in Proposition 5.29. Then b acts injectively on*

$$\bigcup_{n \in \mathbb{N}} \ker(R^n)$$

and therefore is a vector space isomorphism onto its image

$$b\left(\bigcup_{n \in \mathbb{N}} \ker(R^n) \right) = \bigcup_{n \in \mathbb{N}} \ker\left((\mathrm{Id}_L - b\psi)^n \right). \tag{28}$$

So the Hochschild homology of L is given by the kernel and the cokernel of the linear map $\overline{b}$ that results from b after dividing out the subspaces of elements whose reduction sequence in Proposition 5.29 terminates at zero. With the notation from Remark 5.30 it is of the *form*

$$\begin{aligned}\bar{b}\colon \bigoplus_{e\in E^1} (r(e)kEs(e)d(e) \oplus s(e)(kE)^* r(e)d(e^*)) \oplus \bigoplus_{v\in X} k\omega_v &\to \bigoplus_{m\in\mathbb{Z}} V_m\\ xd(y) &\mapsto xy - yx.\end{aligned}$$

Proof. For the first claim, let $n \in \mathbb{N}$ and $\eta \in \ker(R^n) \cap \ker(b)$. By assumption we have $\psi b(\eta) = 0$ and iteratively we get

$$\eta = \mathrm{Id}_{\Omega^1(L)_\#}(\eta) - \psi b(\eta) = R\eta = \cdots = R^n\eta = 0.$$

Thus, b acts injectively on $\bigcup_{n\in\mathbb{N}} \ker(R^n)$ and it suffices to verify the equality in (28).
For the $\subseteq$ inclusion, let $\eta \in \ker(R^n)$. The operator R^n can be spelled out with the binomial formula since ψb commutes with the identity and we get that $R^n\eta = 0$ can be rewritten as a fixed point formula:

$$R^n\eta = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (\psi b)^j(\eta) = 0 \quad\iff\quad \eta = -\sum_{j=1}^{n} (-1)^j \binom{n}{j} (\psi b)^j(\eta). \tag{29}$$

If we apply b to this fixed point formula, we arrive at the analogous fixed point formula for elements in $\ker\left((\mathrm{Id}_L - b\psi)^n\right)$ instead:

$$\begin{aligned} & b\eta = -\sum_{j=1}^{n} (-1)^j \binom{n}{j} b(\psi b)^{j-1}\psi b(\eta) = -\sum_{j=1}^{n} (-1)^j \binom{n}{j} (b\psi)^j(b\eta)\\ \iff\quad & b\eta \in \ker\left((\mathrm{Id}_L - b\psi)^n\right).\end{aligned}$$

This shows the $\subseteq$ inclusion. Conversely, for any $a \in \ker\left((\mathrm{Id}_L - b\psi)^n\right)$ the corresponding fixed point formula shows

$$a = -\sum_{j=1}^{n} (-1)^j \binom{n}{j} (b\psi)^j(a) = b\left(-\sum_{j=1}^{n} (-1)^j \binom{n}{j} \psi(b\psi)^{j-1}(a)\right).$$

If we call the element inside the brackets η' with $a = b(\eta')$, then its defining equation shows $\eta' \in \ker(R^n)$ by the equivalence in (29):

$$\eta' = -\sum_{j=1}^{n} (-1)^j \binom{n}{j} \psi(b\psi)^{j-1}b(\eta') = -\sum_{j=1}^{n} (-1)^j \binom{n}{j} (\psi b)^j(\eta').$$

This establishes the inverse $\supseteq$ inclusion in (28). □

In combination with the last theorem, the reduction algorithm brought us yet another step closer to the Hochschild homology of a relative Leavitt path algebra. In fact, it boils the task down to the analysis of the closed path structure of the underlying quiver with an extra emphasis on the fibres for the chosen regular vertices in X.
So far, the reduction algorithm refrained from dealing with closed paths at all since we defined ψ to be zero on them. However, to some extent, the idea of separating an outer edge in Definition 5.28 also works for non-trivial closed paths. To see this, consider the analogous assignment

$$\psi'(\theta) := \theta_1 \cdots \theta_{|m|-1} d(\theta_{|m|}) \tag{30}$$

for a closed path $\theta = \theta_1 \cdots \theta_{|m|} \in \mathcal{C}_{v,m}$ with $m \in \mathbb{Z}\setminus\{0\}$. The corresponding operator

$$\sigma = (\mathrm{Id}_L - b\psi')$$

also shuffles the last edge or ghost edge in front, but this time it causes no simplification. It rather yields a rotated closed path of length $|m|$ that is now based at $s(\theta_{|m|})$ instead of $v = r(\theta_{|m|}) = s(\theta_1)$.

Definition 5.32 ([2, p. 34])**.** Let E be a quiver. Let $m \in \mathbb{Z}\setminus\{0\}$ and let V_m be the corresponding k-vector space of closed paths. The cyclic group of order $|m|$ acts on it by rotation of the edge positions. The generator of the clockwise rotation σ is called the *cyclic permutation operator*. Explicitly, it is the linear map defined by

$$\begin{aligned} \sigma \colon V_m &\to V_m \\ \theta_1 \cdots \theta_{|m|-1}\theta_{|m|} &\mapsto \theta_{|m|}\theta_1 \cdots \theta_{|m|-1}. \end{aligned}$$

For a closed path $\theta = \theta_1 \cdots \theta_{|m|} \in \mathcal{C}_{v,m}$, the set

$$\mathcal{O}_\theta = \{\sigma^j(\theta) \mid 0 \le j < |m|\}$$

is called the *orbit* of θ. If all paths in the orbit are based at different vertices, that is, $|s(\mathcal{O}_\theta)| = |m|$, then θ is called a *cycle*.

Remark 5.33. If started at a closed path $\theta \in \mathcal{C}_{v,m}$, the assignment (30) would have caused periodic sequences in Proposition 5.29 instead. They run through the orbits $\mathcal{O}_\theta$ or $\psi'(\mathcal{O}_\theta)$, respectively. So the reduction idea fails unless there is an exit condition for a highlighted element in the orbit. Nevertheless, ψ' itself serves as an inverse for the zeroth face map $xd(y) \mapsto xy$ and allows to rewrite the domain of $\bar{b}$ in Theorem 5.31 in a more compact form.

Proposition 5.34. *Let $\bar{b}$ be as in Theorem 5.31. Its domain is isomorphic to the following direct sum of vector spaces indexed by $\mathbb{Z}$: For $m \neq 0$ we use (30) to obtain*

$$\begin{aligned}\bigoplus_{e\in E^1} (r(e)kEs(e)d(e) \oplus s(e)(kE)^* r(e)d(e^*)) &\cong \bigoplus_{m\in\mathbb{Z}\setminus\{0\}} V_m\\ xd(y) &\mapsto xy,\\ \psi'(\theta) = \theta_1 \cdots \theta_{|m|-1} d(\theta_{|m|}) &\mapsfrom \theta = \theta_1 \cdots \theta_{|m|}.\end{aligned}$$

For $m = 0$, we introduce $\bigoplus_{x\in X} k\omega_x \cong \bigoplus_{x\in X} kx$. Under these isomorphisms we have that $\bar{b}$ respects the $\mathbb{Z}$-grading and decomposes into $\mathrm{Id}_{V_m} - \sigma\colon V_m \to V_m$ for $m \neq 0$ and

$$\begin{aligned}M\colon \bigoplus_{x\in X} kx &\to V_0 = \bigoplus_{v\in E^0} kv\\ x &\mapsto x - \sum_{f\in s^{-1}(x)} r(f).\end{aligned}$$

Proof. The displayed assignments for basis elements are clearly bijective and therefore provide vector space isomorphisms by construction. Throughout, the composition with $\bar{b}$ does not affect the path length and therefore decomposes into component maps for the m-summands. For $m \neq 0$, it keeps the input and subtracts the rotated path that has the ultimate edge in front. Hence, we have $\mathrm{Id}_{V_m} - \sigma\colon V_m \to V_m$, as claimed.
For $m = 0$, recall that we defined

$$\omega_x = \sum_{f\in s^{-1}(x)} fd(f^*) = -\sum_{f\in s^{-1}(x)} f^* d(f).$$

Hence, M is a direct consequence of (27). □

Remark 5.35. So, intuitively, for the space of m-homogeneous closed paths with $m \neq 0$ the Hochschild homology measures the invariants under the permutation action in its first and the coinvariants in its zeroth degree. On the level of vertices, though, it is not yet clear what the 0-component map M actually does in its current form. However, it can be captured in terms of a $(|E^0| \times |X|)$-matrix with finitely supported columns as well. In this way, the adjacency matrix of the quiver shows up.

Definition 5.36. Let E be a quiver and let $X \subseteq \operatorname{Reg}(E)$ be a set of chosen regular vertices. Then the $(|X| \times |E^0|)$-matrix A_X with entries

$$a_{x,v} := |\{e \in E^1 \colon s(e) = x,\, r(e) = v\}| \in \mathbb{N}_0$$

is called the *X-reduced adjacency matrix* of E. For every index pair $(x, v) \in X \times E^0$ it counts the number of edges from x to v in the quiver. Note that every row is finitely supported and sums up to the cardinality of the fibre of the corresponding regular vertex.

This brings us to the final result of this chapter.

Theorem 5.37 ([2, p. 253], [5, Thm. 4.4])**.** *Let E be a quiver and let $X \subseteq \operatorname{Reg}(E)$ be a set of chosen regular vertices. Let $L = L_k^X(E)$ be the associated relative Leavitt path algebra. Its Hochschild homology decomposes into a direct sum of vector spaces*

$$HH_\bullet(L) = \bigoplus_{m\in\mathbb{Z}} {}_mHH_\bullet(L)$$

and we have for any $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ that

$${}_mHH_n(L) = \begin{cases} \operatorname{coker}(\mathrm{Id}_{V_m} - \sigma \colon V_m \to V_m), & n = 0,\ m \neq 0, \\ \operatorname{coker}(1_X - A_X^T \colon \bigoplus_{x\in X} k \to \bigoplus_{v\in E^0} k), & n = 0,\ m = 0, \\ \ker(\mathrm{Id}_{V_m} - \sigma \colon V_m \to V_m), & n = 1,\ m \neq 0, \\ \ker(1_X - A_X^T \colon \bigoplus_{x\in X} k \to \bigoplus_{v\in E^0} k), & n = 1,\ m = 0, \\ 0, & n \geq 2. \end{cases}$$

Here, σ denotes the cyclic permutation operator from 5.32, while A_X^T and $1_X \in M_{|X|}(k) \subseteq M_{|X|\times|E^0|}(k)$ are the transposed X-reduced adjacency matrix from 5.36 and the embedded identity matrix, respectively.

Proof. For $n \geq 2$, the claim is already established in Theorem 5.17. For $n \in \{0, 1\}$, the $\mathbb{Z}$-decomposition directly follows from Proposition 5.34 and Theorem 5.31. The claim for $m \neq 0$ is also already contained in Proposition 5.34. Finally, for $m = 0$, the linear map M has to be stated in the displayed matrix form. By definition, the column for a regular vertex $x \in X$ is given by the translation of

$$M(x) = x - \sum_{f\in s^{-1}(x)} r(f)$$

in vector notation. If we count the multiplicity of $v \in E^0$ in this expression, we indeed obtain:

$$\begin{aligned} m_{v,x} &= \delta_{v,x} - |\{f : x \to v\}| \\ &= \delta_{v,x} - (A_X)_{x,v} \\ &= (1_X - A_X^T)_{v,x}. \end{aligned}$$

This shows the claim. □

5.2 Periodic cyclic homology

We have already commented on the correspondence between the theory of commutative algebras and classic geometry. It motivates to interpret its generalisation to all algebras as some sort of noncommutative geometry. In this correspondence, the notion of a quasi-free algebra is the correct generalisation for the geometry of a smooth variety in the commutative framework. By this means, quasi-free algebras capture the noncommutative geometry that extends smooth varieties, which in turn are a quite restrictive family of smooth manifolds.

Classically, for any smooth manifold it is known that its de Rham cohomology encodes crucial information about its structure in algebraic language. Which homology theory suits as the correct generalisation to all algebras? For this purpose, the periodic cyclic homology $HP_\bullet$ was introduced in the 1980s and 1990s and intensely studied since then. In general, its construction is fairly complicated and relies on the interplay of the universal derivation d and the Hochschild boundary b in the framework of higher noncommutative forms. See also [20] or [24].

However, in the restrictive case of quasi-free algebras it merely boils down to the homology theory of the so called X-complex. It was introduced by Quillen in [25] and later used in his collaboration with Cuntz in [16] to use the theory of quasi-free algebras as a starting point for the more challenging general case. See also [14] for more details.

Definition 5.38 ([16, p. 20], [14, p. 6])**.** Let A be an algebra. Define the two vector spaces

$$X(A)_+ := A, \quad X(A)_- := \Omega^1(A)_\#.$$

We consider two linear maps between them. Firstly,

$$\begin{aligned} \#d\colon A &\to \Omega^1(A)_\#,\\ a &\mapsto d(a) + [A, \Omega^1(A)_\#], \end{aligned}$$

as composition of the universal derivation with the commutator quotient map. Secondly,

$$\begin{aligned} b\colon \Omega^1(A)_\# &\to A,\\ xd(y) + [A, \Omega^1(A)_\#] &\mapsto [x, y], \end{aligned}$$

as in Definition 5.17. They clearly satisfy $b \circ \#d = 0$ and also $\#d \circ b = 0$ because of $d([x, y]) = [x, dy] - [y, dx]$. Therefore, they constitute a two-periodic chain complex

$$X(A)\colon\ X(A)_+ \underset{b}{\overset{\#d}{\rightleftarrows}} X(A)_-$$

with $X(A)_+$ in even and $X(A)_-$ in odd degree. This complex is called the *X-complex* of A.

The main reason why the X-complex is relevant in our context of relative Leavitt path algebras is the fact that its homology computes the periodic cyclic homology in the quasi-free case.

Theorem 5.39 ([16, p. 20 ff.])**.** *Let L be a relative Leavitt path algebra and let $X(L)$ be its X-complex as in Definition 5.38, which is not to be confused with the chosen set of regular vertices in the construction of L. Let $HP_\bullet$ denote the periodic cyclic homology of an algebra. Then we have*

$$H_\bullet(X(L)) \cong HP_\bullet(L).$$

Spelled out, this means

$$\begin{aligned} HP_0(L) &\cong \frac{\ker(\#d\colon L \to \Omega^1(L)_\#)}{\operatorname{Im}(b\colon \Omega^1(L)_\# \to L)},\\ HP_1(L) &\cong \frac{\ker(b\colon \Omega^1(L)_\# \to L)}{\operatorname{Im}(\#d\colon L \to \Omega^1(L)_\#)}. \end{aligned}$$

Next we compute the homology of the X-complex for a relative Leavitt path algebra L. For this, the computations of the Hochschild homology can be recycled.

Proposition 5.40. *Let L be a relative Leavitt path algebra and let $\#d$ and b be as in Definition 5.38. Then $\#d$ induces a linear map*

$$\overline{\#d}\colon HH_0(L) \to HH_1(L)$$

on its Hochschild homology with

$$\begin{aligned}HP_0(L) &\cong \ker(\overline{\#d}\colon HH_0(L) \to HH_1(L)),\\ HP_1(L) &\cong \operatorname{coker}(\overline{\#d}\colon HH_0(L) \to HH_1(L)).\end{aligned}$$

Proof. The two identities $b \circ \#d = 0$ and $\#d \circ b = 0$ show that $\#d$ induces a linear map

$$\overline{\#d}\colon L/\mathrm{Im}(b) \cong HH_0(L) \to \ker(b) \cong HH_1(L).$$

Its kernel and cokernel compute the homology of the X-complex by design and hence Theorem 5.39 gives the result:

$$\begin{aligned}HP_0(L) &\cong \frac{\ker(\#d)}{\mathrm{Im}(b)} = \ker(\overline{\#d}),\\ HP_1(L) &\cong \frac{\ker(b)}{\mathrm{Im}(\#d)} = \frac{HH_1(L)}{\mathrm{Im}(\overline{\#d})} = \operatorname{coker}(\overline{\#d}).\end{aligned}$$ □

Remark 5.41. Note that all vertices $v \in E^0$ already satisfy $\#d(v) = 0$ since $vd(v)v = 0$ in (15) implies:

$$\begin{aligned}d(v) &= vd(v) + d(v)v\\ &= [v, vd(v) - d(v)v].\end{aligned}$$

In particular, the induced map $\overline{\#d}$ annihilates the entire subspace of 0-homogeneous elements in the commutator quotient $L_\#$.

To analyse the effect of $\overline{\#d}$ on m-homogeneous elements for $m \neq 0$, we should choose a fundamental domain for the permutation action on V_m generated by σ. All closed paths in the same orbit represent the same commutator class, but, a priori, there is no preferred representative in it. If we parametrise its elements and fix an artificial ordering of the visited vertices and fibres, however, then we are in position to address one.

Definition 5.42 ([2, p. 254]). Let E be a quiver. Fix an enumeration of its vertices as well as a subdominant enumeration for any fibre $s^{-1}(v)$ with $v \in E^0 \setminus \mathrm{Sink}(E)$. Let $\theta = \theta_1 \cdots \theta_{|m|} \in \mathcal{C}_{v,m}$ be a closed path for some $m \in \mathbb{Z} \setminus \{0\}$. The sum over all distinct elements in the orbit

$$N(\theta) := \sum_{\zeta \in \mathcal{O}_\theta} \zeta$$

is called the *norm* of θ. Since the following concepts only depend on the path length, we can assume without loss of generality that m is positive. Define the *order* of θ as the cardinality of its orbit

$$\mathrm{ord}(\theta) := \min\{j \in \mathbb{N} \colon \sigma^j(\theta) = \theta\} \mid m$$

and the *multiplicity* of θ as the codivisor $\mathrm{mp}(\theta) := m'$ with $m = \mathrm{ord}(\theta) m'$. Correspondingly, the closed subpath

$$\theta^{(r)} := \theta_1 \cdots \theta_{\mathrm{ord}(\theta)} \in \mathcal{C}_{v,\mathrm{ord}(\theta)}$$

is called the *radical* of $\theta = (\theta^{(r)})^{\mathrm{mp}(\theta)}$. It is designed to have pairwise different permutations $\sigma^j(\theta^{(r)}) \in \mathcal{O}_{\theta^{(r)}}$ that parametrise the orbit:

$$\mathcal{O}_{\theta^{(r)}} \cong \mathcal{O}_\theta, \quad x \mapsto x^{\mathrm{mp}(\theta)}.$$

To pick a unique maximal representative in $\mathcal{O}_{\theta^{(r)}}$, use the following algorithm:

- Pick the maximum $v_{\max}$ of all occurring base vertices and discard all permutations based at different vertices. If $\theta^{(r)}$ is a cycle, this already determines a unique permutation in the orbit.

- Pick the maximum $e_{\max}$ among all candidates for the leading edge $s^{-1}(v_{\max}) \cap \{\theta_j \mid 1 \leq j \leq \mathrm{ord}(\theta)\}$ and discard all permutations with different leading edge.

- Proceed with the descending ordering of the remaining permutations at the next edge until only one maximal permutation is left.

The corresponding closed path $\theta^c \in \mathcal{O}_\theta$ is called the *orbit representative* of θ. The set of all orbit representatives is called Θ_m and forms a fundamental domain for the permutation action.

Proposition 5.43. *Let $m \in \mathbb{Z} \setminus \{0\}$ and let $\chi \colon V_m \to V_m$ be the linear map that is defined* *by annihilating the orbit representatives for closed paths $\theta \in \mathcal{C}_{v,m}$:*

$$\theta \mapsto \begin{cases} \theta, & \theta \neq \theta^c, \\ 0, & \theta = \theta^c. \end{cases}$$

For any closed path, we have that the sequence

$$((\mathrm{Id}_{V_m} - (\mathrm{Id}_{V_m} - \sigma) \circ \chi)^n \theta)_{n \in \mathbb{N}_0} \tag{31}$$

becomes constant after at most $\mathrm{ord}(\theta)$ *iterations with limiting word* θ^c*. Likewise, the sequence*

$$((\mathrm{Id}_{V_m} - \chi \circ (\mathrm{Id}_{V_m} - \sigma))^n \theta)_{n \in \mathbb{N}_0} \tag{32}$$

also becomes constant after at most $\mathrm{ord}(\theta) - 1$ *iterations. Here, the final expression is* 0 *unless we started with* $\theta = \theta^c$*. In this case, it is the norm* $N(\theta)$*.*

Proof. By design, the first operator in (31) acts as follows:

$$(\mathrm{Id}_{V_m} - (\mathrm{Id}_{V_m} - \sigma) \circ \chi)(\theta) = \begin{cases} \sigma(\theta), & \theta \neq \theta^c, \\ \theta^c, & \theta = \theta^c. \end{cases}$$

This already shows the first claim. The second operator in (32) distinguishes the following cases:

$$\begin{aligned} (\mathrm{Id}_{V_m} - \chi \circ (\mathrm{Id}_{V_m} - \sigma))(\theta) &= \theta - \chi(\theta) + \chi(\sigma(\theta)) \\ &= \begin{cases} \theta = N(\theta), & \mathrm{ord}(\theta) = 1, \\ \theta^c + \sigma(\theta^c), & \mathrm{ord}(\theta) > 1,\ \theta = \theta^c, \\ 0, & \mathrm{ord}(\theta) > 1,\ \theta = \sigma^{-1}(\theta^c), \\ \sigma(\theta), & \text{otherwise}. \end{cases} \end{aligned}$$

So it remains to show the claim for non-trivial orbits $\mathrm{ord}(\theta) > 1$. The sequence started at $\sigma(\theta^c)$ runs through all permutations up to $\sigma^{\mathrm{ord}(\theta)-1}(\theta^c)$, where it eventually becomes zero. Finally, for θ^c itself, the sequence gains an additional summand $\sigma^j(\theta^c)$ after the j-th iteration until $j = \mathrm{ord}(\theta) - 1$. Then the third case applies and the sequence remains $N(\theta)$ afterwards. This shows the claim. $\square$

Corollary 5.44. *Let L be a relative Leavitt path algebra and let E be the underlying quiver. Let $m \in \mathbb{Z}\backslash\{0\}$ and let Θ_m denote the set of orbit representatives as in Definition 5.42. Then the m-homogeneous subspaces of the Hochschild homology of L take the following shape:*

$$
\begin{aligned}
{}_mHH_0(L) &\cong \bigoplus_{\theta^c \in \Theta_m} \theta^c k, \\
{}_mHH_1(L) &\cong \bigoplus_{\theta^c \in \Theta_m} N(\theta^c) k.
\end{aligned}
$$

Proof. The argument in Theorem 5.31 applies to the pair $\mathrm{Id}_{V_m} - \sigma$ and χ from Proposition 5.43 in the role of b and ψ, respectively. It shows that $\mathrm{Id}_{V_m} - \sigma$ acts injectively on the subspace of those closed paths whose second reduction sequence (32) becomes zero. Hence, we may pass from $\mathrm{Id}_{V_m} - \sigma$ to the corresponding quotient complex without changing its kernel or cokernel. By Proposition 5.43, we also know that the induced map takes the following form:

$$
\begin{aligned}
\overline{\mathrm{Id}_{V_m} - \sigma} \colon \bigoplus_{\theta^c \in \Theta_m} N(\theta^c) k &\to \bigoplus_{\theta^c \in \Theta_m} \theta^c k \\
N(\theta^c) &\mapsto N(\theta^c) - \sigma N(\theta^c).
\end{aligned}
$$

However, since the norm is invariant under σ by construction, this is the zero map. Thus, the domain and the range coincide with its kernel and cokernel. A comparison with Theorem 5.37 now shows the claim. □

Since it relies on the non-canonical choices of the orbit representatives in Definition 5.42, this formulation of the Hochschild homology is in some sense inferior to the one in Theorem 5.37. However, it allows to analyse the induced map $\overline{\#d}$, which encodes the periodic cyclic homology, in a more convenient way.

Proposition 5.45. *Let the notation be as in Proposition 5.40. For any $m \in \mathbb{Z}$, we have that $\overline{\#d}$ maps m-homogeneous elements to m-homogeneous elements. Thus, the corresponding component maps*

$$
{}_mD \colon {}_mHH_0(L) \to {}_mHH_1(L)
$$

cause a decomposition of the periodic cyclic homology into

$$\begin{aligned} HP_\bullet(L) &= \bigoplus_{m\in\mathbb{Z}} {}_mHP_\bullet(L), \\ {}_mHP_0(L) &\cong \ker({}_mD), \\ {}_mHP_1(L) &\cong \operatorname{coker}({}_mD). \end{aligned}$$

Concretely, we have ${}_0D = 0$ *and if we use the isomorphisms in Corollary 5.44, then we get the following for* $m \neq 0$*:*

$$\begin{aligned} {}_mD\colon \bigoplus_{\theta^c\in\Theta_m} \theta^c k &\to \bigoplus_{\theta^c\in\Theta_m} N(\theta^c)k \\ \theta^c &\mapsto \operatorname{mp}(\theta^c)N(\theta^c). \end{aligned}$$

Proof. It suffices to check the claimed shape of the maps ${}_mD$. Then the remaining part follows from Proposition 5.40. In fact, ${}_0D = 0$ is already established in the subsequent Remark 5.41. For $m \neq 0$ and a closed path $\theta = \theta_1 \cdots \theta_{|m|} \in \mathcal{C}_{v,m}$, the essential observation is the following computation in $\Omega^1(L)_\#$:

$$\begin{aligned} \#d(\theta) &= d(\theta) + [L, \Omega^1(L)] \\ \text{[Leibniz]} &= \sum_{j=0}^{|m|-1} \theta_1 \cdots d(\theta_{|m|-j}) \cdots \theta_{|m|} + [L, \Omega^1(L)] \\ \text{[shift edges]} &= \sum_{j \bmod (|m|)} \theta_{|m|-j+1} \cdots \theta_{|m|}\theta_1 \cdots d(\theta_{|m|-j}) + [L, \Omega^1(L)] \\ &= \sum_{j \bmod (|m|)} \psi'(\sigma^j(\theta)) \\ &= \psi'\left(\sum_{j\,(|m|)} \sigma^j(\theta)\right) \\ &= \psi'\left(\operatorname{mp}(\theta)N(\theta)\right). \end{aligned}$$

Since ψ' from Proposition 5.34 translates between closed paths and their representation in $\Omega^1(L)_\#$, this immediately implies the claimed form of ${}_mD$ on the orbit representatives. □

Remark 5.46. The last result shows that the subspaces ${}_0HH_\bullet(L)$ of 0-homogeneous elements pass on to the periodic cyclic homology completely. Moreover, for m-homogeneous elements with $m \neq 0$, the discussion boils down to multiplication maps over k. Their behaviour for any summand depends on the multiplicity $\mathrm{mp}(\theta^c) \,|\, m$ of the involved orbit representative θ^c, that is, the codivisor of its order introduced in Definition 5.42. In fact, $\mathrm{mp}(\theta)$ is either invertible or zero over k. So ${}_mD$ annihilates all orbit representatives with multiplicity $\mathrm{mp}(\theta) \in \mathrm{char}(k)\,\mathbb{Z}$ and is invertible for the remaining summands. To sum up, we have:

$$\ker(\mathrm{mp}(\theta)\cdot()\colon k \to k) = \mathrm{coker}(\mathrm{mp}(\theta)\cdot()\colon k \to k) = \begin{cases} k, & \mathrm{mp}(\theta) \in \mathrm{char}(k)\,\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5.47. *Let $L = L_k^X(E)$ be a relative Leavitt path algebra. Then its periodic cyclic homology for $m \in \mathbb{Z}$ and $n = 0, 1$ is given by*

$${}_mHP_n(L) = \begin{cases} \bigoplus_{\theta^c \in \Theta_m \colon \mathrm{char}(k)\,|\,\mathrm{mp}(\theta^c)} k, & n = 0,\ m \neq 0, \\ \mathrm{coker}(1_X - A_X^T\colon \bigoplus_{x \in X} k \to \bigoplus_{v \in E^0} k), & n = 0,\ m = 0, \\ \bigoplus_{\theta^c \in \Theta_m \colon \mathrm{char}(k)\,|\,\mathrm{mp}(\theta^c)} k, & n = 1,\ m \neq 0, \\ \ker(1_X - A_X^T\colon \bigoplus_{x \in X} k \to \bigoplus_{v \in E^0} k), & n = 1,\ m = 0. \end{cases}$$

In particular, if the characteristic of the base field is zero, then only the 0-homogeneous parts remain and coincide with those of the Hochschild homology ${}_0HH_\bullet(L)$.
If the characteristic is non-zero, that is, a prime $p = \mathrm{char}(k)$, then there are still contributions in the m-homogeneous parts for $p|m$. They come from those orbit representatives that arise as p-th powers.

Proof. For $m = 0$, the claim follows from ${}_0D = 0$ in Proposition 5.45 and Theorem 5.37. For $m \neq 0$, Proposition 5.45 applies again and leaves to compute the kernel and the cokernel of the diagonal multiplication map

$$\bigoplus_{\theta^c \in \Theta_m} \mathrm{mp}(\theta^c)\cdot()\colon \bigoplus_{\theta^c \in \Theta_m} k \to \bigoplus_{\theta^c \in \Theta_m} k.$$

By the discussion in Remark 5.46 they consist of those summands, for which the multiplicity $\mathrm{mp}(\theta^c)$ is a multiple of the characteristic of the base field k. This shows the claim. $\square$

To conclude this thesis, we apply these homological computations to the examples introduced in Section 3.1.

Example 5.48. Recall the algebraic Cuntz algebras $L_k(R_n)$ from Example 3.9 for $n \in \mathbb{N}$. When it comes to the m-homogeneous parts of the Hochschild homology in degree 0 or 1 with $m \neq 0$, then Corollary 5.44 tells us that we obtain a k-summand for every orbit representative in Θ_m. Note that every path in R_n is closed by force and based at 1. If we decide for the suggestive ordering

$$e_1 < \cdots < e_n$$

of the fibre $s^{-1}(1) = R_n^1$, then the orbit representatives of length $|m|$ can be counted systematically by using combinatorial arguments for $|m|$-tuples in n letters.
In the 0-homogeneous case, the discussion breaks down to the (1×1)-matrix $1_X - A_X^T = (1-n)$. If we further assume that k has characteristic zero, say, $k = \mathbb{C}$, then only the kernel and the cokernel of this matrix possibly contribute to the periodic cyclic homology at all.
Since $1-n$ is clearly invertible, unless we are dealing with the Laurent polynomials for $n = 1$, we obtain:

$$HP_l(L_{\mathbb{C}}(R_n)) = {}_0HH_l(L_{\mathbb{C}}(R_n)) = \begin{cases} \delta_{n,1}\ \mathbb{C}, & l = 0, \\ \delta_{n,1}\ \mathbb{C}, & l = 1. \end{cases}$$

For $l = 0$, the contribution for the Laurent polynomials $\mathbb{C}[x, x^{-1}]$ should not come as a surprise since it is a commutative algebra with

$$HH_0(\mathbb{C}[x, x^{-1}]) = \mathbb{C}[x, x^{-1}]_\# = \mathbb{C}[x, x^{-1}] = \bigoplus_{m \in \mathbb{Z}} x^m\, \mathbb{C}\,.$$

For $l = 1$, however, the $\mathbb{C}$-summand has a more interesting background. Tracing the chain of isomorphisms back, it comes from the computation (27) that motivated to introduce the noncommutative form

$$\omega_1 = \sum_{f \in s^{-1}(1)} f d(f^*) = x d(x^{-1})$$

in Remark 5.30. It spans the 0-homogeneous part of $\ker(b) = HH_1(\mathbb{C}[x, x^{-1}])$ because of

$$b(\omega_1) = x x^{-1} - x^{-1} x = 0.$$

On the other hand, if we allow for infinitely many loops based at the only vertex 1, then R_∞ no longer admits a regular vertex. In fact, the 0-homogeneous discussion is now concerned with a linear map from the zero vector space into the one-dimensional vector space indexed by $\{1\} = R_\infty^0$. Consequently, $0\colon \{0\} \to k$ admits a trivial kernel and a one-dimensional cokernel, just as in the base-field discussion in Example 5.11:

$$ {}_0HP_l(L_k(R_\infty)) = {}_0HH_l(L_k(R_\infty)) = \begin{cases} k, & l = 0, \\ 0, & l = 1. \end{cases} $$

Example 5.49. For $n \in \mathbb{N}$ and the matrix algebras $L_k(A_n) \cong M_n(k)$ in Example 3.10, there is no closed path of positive length at all. The quiver

$$ A_n = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n} $$

is an instance of an *acyclic* quiver as in Definition 5.27. Consequently, the m-homogeneous discussion for $m \neq 0$ drops out entirely and both the Hochschild homology and the periodic cyclic homology agree. Since v_n is a sink, one last row of zeros is discarded and the X-reduced adjacency matrix only admits 1-entries on the first upper diagonal. Hence, the transposed matrix of interest in the 0-homogeneous case has an $n \times (n-1)$-shape and looks like:

$$ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}. $$

It has a maximal rank of $n-1$ and therefore a trivial kernel and a cokernel of dimension 1. Thus, the Hochschild homology agrees with the one of the base field in Example 5.11 and we have

$$ HP_l(M_n(k)) = HH_l(M_n(k)) = \begin{cases} k, & l = 0, \\ 0, & l \neq 0. \end{cases} $$

Example 5.50 ([2, p. 255]). Finally, we have met two possibilities to capture the algebraic Toeplitz algebra $k[s, s^* \mid x^*x = 1]$ from Example 3.11 in terms of a relative Leavitt path algebra. Either as the Cohn algebra $C(R_1)$ or as the Leavitt path algebra $L_k(E_T)$. In either way, the m-homogeneous discussion for $m \neq 0$ leads to precisely one orbit representative

$$\theta_m^c = \begin{cases} s^m, & m > 0, \\ (s^*)^{-m}, & m < 0. \end{cases}$$

It is the m-th power of the unique loop in R_1 or E_T and yields

$${}_{(m\neq 0)}HH_n(k[s, s^* \mid x^* x = 1]) = \delta_{n \leq 1}\, k.$$

In particular, with an eye on its implications for HP, we have a maximal multiplicity $\mathrm{mp}(\theta_m^c) = |m|$ for the singleton $\{\theta_m^c\} = \Theta_m$ all the time.
For the 0-homogeneous case and the quiver E_T, we are confronted with a reduced adjacency matrix (1, 1). Thus, the (2×1)-matrix of interest looks like

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} : k \to k^2.$$

It has maximal rank and therefore again a trivial kernel and a one-dimensional cokernel. Of course, we could have arrived there by looking at R_1 and the empty set of chosen vertices as well. In total, we have:

$${}_mHP_n(k[s, s^* \mid x^* x = 1]) = \begin{cases} \delta_{\mathrm{char}(k)|m}\, k, & n = 0,\ m \neq 0, \\ k, & n = 0,\ m = 0, \\ \delta_{\mathrm{char}(k)|m}\, k, & n = 1,\ m \neq 0, \\ 0, & n = 1,\ m = 0. \end{cases}$$

References

[1] Gene Abrams. “Leavitt path algebras: The first decade”.
In: *Bull. Math. Sci. 5, (2015)*, pp. 59–120.

[2] Gene Abrams, Pere Ara, Mercedes Siles Molina. *Leavitt Path Algebras*.
Springer-Verlag, 2017.

[3] Pere Ara, Miquel Brustenga. “The regular algebra of a quiver”.
In: *J. Algebra 309, (2007)*, pp. 207–235.

[4] Pere Ara, Miquel Brustenga.
“Module theory over Leavitt path algebras and K-theory”.
In: *J. Pure Appl. Algebra 214, (2010)*, pp. 1131–1151.

[5] Pere Ara, Guillermo Cortiñas. “Tensor products of Leavitt path algebras”.
In: *Proc. Amer. Math. Soc. 141, (2013)*, pp. 2629–2639.

[6] Pere Ara, Miquel Brustenga, Guillermo Cortiñas. “K-theory of Leavitt path algebras”.
In: *Münster J. of Math. 2, (2009)*, pp. 5–34.

[7] Bruce Blackadar. *K-theory for operator algebras*.
MSRI publications, second edition, 1998.

[8] Alexey Bondal, Ilya Zhdanovskiy. “Coherence of relatively quasi-free algebras”.
In: *EJM 1, (2015)*, pp. 695–703.

[9] Xiao-Wu Chen, Huanhuan Li, Zhengfang Wang.
An explicit projective bimodule resolution of a Leavitt path algebra.
In: (2017). arXiv: 1711.01876v1.

[10] Xiao-Wu Chen, Huanhuan Li, Zhengfang Wang.
Leavitt path algebras, B_∞-algebras and Keller's conjecture for singular Hochschild cohomology.
In: (2020). arXiv: 2007.06895v2.

[11] Claude Cibils et al. “Cohomology of split algebras and of trivial extensions”.
In: *Glasgow Math. J. 45, (2003)*, pp. 21–40.

[12] Paul Moritz Cohn. *Introduction to Ring Theory.*
Springer-Verlag, Third Printing 2004.

[13] Joachim Cuntz, Ralf Meyer, Jonathan Rosenberg.
Topological and Bivariant K-Theory.
Birkhäuser Verlag AG, 2007.

[14] Joachim Cuntz. "Quillen's work on the foundations of cyclic cohomology".
In: *Journal of K-theory, 11 (3), (2013)*, pp. 559-574.

[15] Joachim Cuntz, Daniel Quillen. "Algebra extensions and nonsingularity".
In: *J. Amer. Math. Soc. 8, (1995)*, pp. 251–289.

[16] Joachim Cuntz, Daniel Quillen. "Cyclic homology and nonsingularity".
In: *J. Amer. Math. Soc. 8, (1995)*, pp. 373–442.

[17] Victor Ginzburg. *Lectures on Noncommutative Geometry.*
In: (2005). arXiv: 0506603v1.

[18] Saunders Mac Lane. *Homology.*
Springer-Verlag, Fourth Printing 1994.

[19] Saunders Mac Lane. "Structure in Mathematics".
In: *Philos. Math. 4, (1996)*, pp. 174–183.

[20] Jean-Louis Loday. *Cyclic Homology.*
Springer-Verlag, First Printing 1992.

[21] Ralf Meyer. "Smooth and rough modules over self-induced algebras".
In: *Münst. J. Math. (2011)*, pp. 29–52.

[22] Ralf Meyer. *Leavitt path algebra notes.*
In: (2020). Handwritten notes.

[23] Ralf Meyer. *Bicategories.*
In: (2020).

[24] Ralf Meyer. *Lecture notes: Noncommutative geometry.*
In: (2020).

[25] Daniel Quillen. “Algebra cochains and cyclic cohomology”. In: *Publications mathématiques de l’I.H.É.S., tome 68 (1988)*, pp. 139–174.

[26] Daniel Quillen. *Lecture notes: Cyclic cohomology and Karoubi operators.* In: (1991).