Mathematisches Institut
Georg-August-Universität Göttingen
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Exercise sheet 1.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Friday, 25.10.2024, 16:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Let $\mathcal{H}$ be a Hilbert space. For a bounded operator $T\colon \mathcal{H} \to \mathcal{H}$, its adjoint is the unique bounded operator T^* that satisfies $\langle T\psi \mid \phi\rangle = \langle \psi \mid T^*\phi\rangle$ for all $\psi, \phi \in \mathcal{H}$. It is known from functional analysis that such an operator always exists. For a densely defined, unbounded operator $T\colon \mathcal{H} \supseteq \operatorname{dom} T \to \mathcal{H}$, the adjoint is the unbounded operator with domain all $\phi \in \mathcal{H}$ for which a vector $\eta = T^*\phi \in \mathcal{H}$ with $\langle T\psi \mid \phi\rangle = \langle \psi \mid \eta\rangle$ for all $\psi \in \operatorname{dom} T$ exists. The operator is called selfadjoint if $T = T^*$.

Exercise 1. For bounded operators, show the properties $(T^*)^* = T$, $(ST)^* = T^*S^*$ for all bounded operators, and $(T^{-1})^* = (T^*)^{-1}$ for all invertible bounded operators.

Exercise 2. An operator on a Hilbert space $T\colon \mathcal{H} \to \mathcal{H}$ is called a *projection* if $T^* = T$ and $T^2 = T$. It is called an *involution* if $T^* = T$ and $T^2 = 1$.

1. Show that the map $T \mapsto 2T - 1$ is a bijection from the set of projections to the set of involutions on $\mathcal{H}$.
2. Let P be a projection. Show that the kernel $\ker P$ and the image $\operatorname{im} P$ are orthogonal and that their (direct) sum is all of $\mathcal{H}$. Show that P is the orthogonal projection onto $\operatorname{im} P$.

Exercise 3. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a possibly unbounded measurable function. Define a densely defined unbounded operator M_f of multiplication by f on $L^2(\mathbb{R}^3)$ by letting its domain consist of all $\psi \in L^2(\mathbb{R}^3)$ with $f \cdot \psi \in L^2(\mathbb{R}^3)$, and $M_f(\psi) := f \cdot \psi$, the pointwise product of two functions. Show that this domain is indeed dense as claimed and that M_f is selfadjoint.

We say that a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ is of Schwartz class if it is smooth and all its (partial) derivatives at any order have a decay at infinity faster that polynomials. It means that if P is a polynomial function on $\mathbb{R}^3$ and $\alpha \in \mathbb{N}^3$ then:

$$\lim_{|\xi| \to \infty} P(\xi) \frac{\partial^\alpha f}{\partial x^\alpha}(\xi) = 0.$$

We denote by $\mathscr{S}(\mathbb{R}^3)$ the vector space of Schwartz functions. We admit that functions in $\mathscr{S}(\mathbb{R}^3)$ are integrable (and square integrable as well) and that $\mathscr{S}(\mathbb{R}^3)$ is stable under Fourier transform.

Exercise 4. *This exercise is a presence exercise and is not part of the homework.* Fix $\hbar > 0$. Let $P_1\colon \mathscr{S}(\mathbb{R}^3) \subseteq L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be the momentum operator,

$$P_1\psi(x_1, x_2, x_3) := -\mathrm{i}\hbar \frac{\partial}{\partial x_1} \psi(x_1, x_2, x_3).$$

Let $\mathcal{F}$ denote the Fourier transform, defined on sufficiently regular functions by

$$(\mathcal{F}\psi)(\xi) := \int_{\mathbb{R}^3} \psi(x) \exp(-2\pi \mathrm{i} x \cdot \xi) \,\mathrm{d}x$$

for all $x, \xi \in \mathbb{R}^3$. This operator is known to extend to a unitary operator on $L^2(\mathbb{R}^3)$, with

$$\mathcal{F}^*\psi(x) = \mathcal{F}^{-1}\psi(x) = \int_{\mathbb{R}^3} \psi(x) \exp(2\pi \mathrm{i} x \cdot \xi) \,\mathrm{d}\xi.$$

1. Show that the operator $\mathcal{F}P_1\mathcal{F}^*$ acts on $\psi \in \mathscr{S}(\mathbb{R}^3)$ by $(\mathcal{F}P_1\mathcal{F}^*\psi)(\xi) = 2\pi\hbar\xi_1 \cdot \psi(\xi)$.
2. Show that the graph of $\mathcal{F}P_1\mathcal{F}^*$ in $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ is a dense subspace of the graph of the densely defined selfadjoint operator M_{ξ_1} for the function $\xi \mapsto \xi_1$ on $\mathbb{R}^3$, as defined in the previous exercise. This means that $M_{2\pi\hbar\xi_1}$ is the "closure" of the unbounded operator $\mathcal{F}P_1\mathcal{F}^*$.
3. Deduce that the closure of the graph of the operator P_1 is the graph of a selfadjoint densely defined operator on $L^2(\mathbb{R}^3)$. In brief, the closure of the operator P_1 is a selfadjoint densely defined operator.