MATHEMATISCHES INSTITUT
GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN
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Exercise sheet 2.

Name

Exercise group (tutor's name)

Exercise	1	2	3	Σ
Points				

Deadline: **Friday, 1.11.2024, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. In this exercise we consider the Hamiltonian $H = \sum_{j=1}^{3} \frac{1}{2m} P_j^2$ where P_j, $j = 1, \dots, 3$ are the momentum operators and $m > 0$ is a constant.

1. Check that H can be defined as an unbounded operator on a domain containing the Schwartz space $\mathscr{S}(\mathbb{R}^3)$. Use similar argument as in Exercise 4 of the previous sheet to find a domain on which it is self adjoint.
2. For $t \in \mathbb{R}$, give a meaning to the operator $U_t \coloneqq \exp\left(-\frac{\mathrm{i}t}{\hbar} H\right)$ and show it is unitary.
3. Show the following integral formula for ψ_0 regular enough, say, in $\mathscr{S}(\mathbb{R}^3)$:
$$U_t \psi_0(x) = \int_{\xi \in \mathbb{R}^3} \mathrm{e}^{2\pi \mathrm{i}\left(\langle x, \xi\rangle - \frac{\pi \hbar t}{m} |\xi|^2\right)} \widehat{\psi_0}(\xi)\, \mathrm{d}\xi$$
4. Show that $\psi \colon (t, x) \mapsto U_t \psi_0(x)$ is the solution to the Schrödinger equation, that is,
$$\mathrm{i}\hbar \frac{\partial}{\partial t} \psi = H\psi,$$
with initial condition ψ_0.

Remember that for a commutative C^*-algebra $\mathcal{A}$, the spectrum $\widehat{\mathcal{A}}$ of the algebra is the set of algebra $*$-homomorphisms $\chi \colon \mathcal{A} \to \mathbb{C}$. Endowed with the coarsest topology for which the evaluation maps at elements of $\mathcal{A}$ are continuous functions, this makes $\widehat{\mathcal{A}}$ a locally compact Haussdorff space (compact if $\mathcal{A}$ is unital), such that the evalution maps give an isomorphism $\mathcal{A} \cong \mathrm{C}\left(\widehat{\mathcal{A}}\right)$. You may use this fact below.

Exercise 2. The goal of this exercise is to define the joint spectrum of an n-tuple $A_1, \dots, A_n$ of (pairwise) commuting bounded selfadjoint operators on a Hilbert space $\mathcal{H}$. The result remains true for unbounded, normal, strongly commuting operators, but we limit the exercise to the bounded, selfadjoint case for simplicity. Let $\mathcal{A}$ be the closed subalgebra of $\mathbb{B}(\mathcal{H})$ generated by the identity operator and $A_1, \dots, A_n$.

1. Show that $\mathcal{A}$ is a commutative unital C*-algebra. So $\mathcal{A} \cong \mathrm{C}\left(\widehat{\mathcal{A}}\right)$.
2. Show that the map $\widehat{\mathcal{A}} \to \mathbb{C}^n, \chi \mapsto (\chi(A_j))_{1 \le j \le n}$, is a homeomorphism onto a compact non-empty subset of $\mathbb{C}^n$. By definition, this image set is the *joint spectrum* of the operators.
3. Assume now that $\mathcal{H}$ is finite-dimensional, so that the A_i are commuting, selfadjoint matrices. Show that the joint spectrum is the set of joint eigenvalues, that is, $(\lambda_j)_{1 \le j \le n} \in \mathbb{C}^n$ for which there is a $v \in \mathcal{H} \setminus \{0\}$ with $A_j v = \lambda_j v$ for all $1 \le j \le n$.

Exercise 3. Consider the momentum operators on $L^2(\mathbb{R}^n)$: $P_j \coloneqq -\mathrm{i}\hbar \frac{\partial}{\partial x^j}, 1 \le j \le n$. They are known to be selfadjoint and to commute strongly. Thus they generate a strongly continuous representation of $\mathbb{R}^n$: $U \colon \mathbb{R}^n \to \mathcal{B}(\mathbb{R}^n)$, $k \mapsto \exp(\sum_{j=1}^{n} \mathrm{i} k_j P_j)$. Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R}^n)$.

1. Show that $\mathcal{F} U(k) \mathcal{F}^*$ acts as a multiplication operator by an explicit function.
2. Conclude that $U(k)\psi(x) = \psi(x + \hbar k)$ for $\psi \in L^2(\mathbb{R}^n)$, $x, k \in \mathbb{R}^n$.
3. For $\psi \in \mathscr{S}(\mathbb{R}^n)$, check that
$$\lim_{k_j \to 0} \frac{1}{\mathrm{i} k_j} (U(0, \dots, k_j, \dots 0)\psi - \psi) = P_j \psi.$$
This is part of Stone's Theorem.