Mathematisches Institut
Georg-August-Universität Göttingen
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Exercise sheet 3.

Name

Exercise group (tutor's name)

Deadline: **Friday, 8.11.2024, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. (This exercise shows that the canonical commutation relations cannot be realised by bounded operators.) In this exercise, we consider a Hilbert space $\mathcal{H}$ with a dense subspace $D \subseteq \mathcal{H}$ and linear maps $X, P\colon D \to D$ with the property that their commutator $[X, P] := XP - PX$ is $\mathrm{i}\hbar$ (times the identity operator on D) with some constant $\hbar > 0$. If you like, you may simplify by pretending that $\hbar = 1$.

1. Show that the position and momentum operators X, P on $L^2(\mathbb{R})$ satisfy this relation with $D = \mathrm{C}_c^\infty(\mathbb{R})$.
2. For $n \in \mathbb{N}$, compute $[X, P^n]$ (use that taking the commutator with X is a derivation).
3. Show that X and P cannot both be continuous.
4. Show that $\mathcal{H}$ cannot be finite-dimensional.

Recall that the algebra of quaternions $\mathbb{H}$ is a 4-dimensional real vector space generated by $(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$ with product generated by the relations: $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$. They form a non-commutative field. The conjugate of an element $h = x + y\mathrm{i} + z\mathrm{j} + t\mathrm{k}$ is the element $\bar{h} = x - y\mathrm{i} - z\mathrm{j} - t\mathrm{k}$. We have:

$$N(h) := \bar{h}h = h\bar{h} = x^2 + y^2 + z^2 + t^2 \in \mathbb{R}_{\geq 0}$$

The map $N\colon \mathbb{H} \to \mathbb{R}$ is a positive definite quadratic form for which $(1, \mathrm{i}, j, k)$ forms an orthonormal basis.

Exercise 2. (The goal of this exercise is to prove that $\mathrm{SU}(2) \,/\, \{\pm \mathrm{Id}_2\} \cong \mathrm{SO}(3)$.) For a quaternion $h \in \mathbb{H}$ as above, we construct the matrix $M_h := \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$ with $a = x + \mathrm{i}y$, $b = z - \mathrm{i}t$.

1. Check that $\mathbb{H} \to \mathbb{M}_2(\mathbb{C})$, $h \mapsto M_h$ is an injective algebra homomorphism, and that for $h \in \mathbb{H}$, we have $M_h^* = M_{\bar{h}}$.

   We now see $\mathbb{H}$ as a subalgebra of matrices. For $g \in \mathrm{SU}(2)$, define a map $\varphi_g\colon \mathbb{H} \to \mathbb{H}$, $h \mapsto ghg^{-1}$.
2. Check that φ_g is an algebra homomorphism and preserves the quadratic form N.
3. Show that there is a map $\varphi\colon \mathrm{SU}(2) \to \mathrm{O}(\mathbb{I})$, where $\mathbb{I}$ is the subspace of $\mathbb{H}$ generated by $\mathrm{i}, \mathrm{j}, \mathrm{k}$ endowed with the inner product corresponding to N. Show that the image of this group homomorphism lies in $\mathrm{SO}(\mathbb{I}) \cong \mathrm{SO}(3)$.
4. Show that $\varphi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective (one could find a preimage for each rotation). Compute the kernel of φ.
5. Conclude that there is an isomorphism $\mathrm{SU}(2) \,/\, \{\pm \mathrm{Id}_2\} \cong \mathrm{SO}(3)$.