Mathematisches Institut
Georg-August-Universität Göttingen
Prof. Dr. Ralf Meyer and Dr. Clément Cren.

Exercise sheet 4.

Name

Exercise group (tutor's name)

Exercise	1	2	3	Σ
Points				

Deadline: **Friday, 15.11.2024, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let $\Theta\colon \mathcal{H} \to \mathcal{H}$ be a time-reversal symmetry on a Hilbert space $\mathcal{H}$. Let $\mathcal{A}$ be the $\mathbb{R}$-subalgebra of $\mathbb{R}$-linear operators on $\mathcal{H}$ generated by Θ and i.

1. If $\Theta^2 = -1$, find an isomorphism between $\mathcal{A}$ and the algebra of quaternions $\mathbb{H}$.
2. If $\Theta^2 = +1$ find an isomorphism between $\mathcal{A}$ and the $\mathbb{R}$-algebra of real 2×2-matrices $\mathbb{M}_2(\mathbb{R})$.

Exercise 2. Let $\pi_0\colon G \to \mathrm{U}(\mathcal{H}) \,/\, \mathrm{U}(1)$ be a projective representation of a group G on a Hilbert space $\mathcal{H}$. Consider any map $\pi\colon G \to \mathrm{U}(\mathcal{H})$ that lifts π_0. Show that there is a map $c\colon G \times G \to \mathrm{U}(1)$ such that:

$$\forall g, h \in G, \pi(g)\pi(h) = c(g,h)\pi(gh).$$

Show that this map is a $\mathrm{U}(1)$-cocycle, that is:

$$\forall g, h, k \in G, c(gh,k)c(g,h) = c(g,hk)c(h,k).$$

Any other lift of π_0 would be of the form $f\pi$ where $f\colon G \mapsto \mathrm{U}(1)$ is any function. Compute the cocycle of $f\pi$ in terms of c, the one of π.

Exercise 3. Consider the position and momentum operators on $L^2(\mathbb{R}^3)$: $X_1, X_2, X_3, P_1, P_2, P_3$. We obtain a projective representation of $\mathbb{R}^6$ by exponentiation:

$$\begin{aligned} \mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{B}(L^2(\mathbb{R}^3)) \\ (s,t) &\mapsto \exp(\mathrm{i}sX)\exp(\mathrm{i}tP). \end{aligned}$$

Compute the cocycle of this projective representation. You may use the computations of $\exp(\mathrm{i}sX)$ and $\exp(\mathrm{i}tP)$ as multiplication and translation operators in the lectures.