Exercise sheet 5.

Name

Exercise group (tutor's name)

Exercise	1	2	3	Σ
Points				

Deadline: **Friday, 22.11.2024, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. *(Another point of view on the SSH model.)* We may understand the SSH model as follows. Put $\Lambda_0 := \frac{1}{2}\mathbb{Z}$ and take the Hilbert space $\mathcal{H} := \ell^2(\Lambda_0)$. Let H be the Hamiltonian on $\mathcal{H}$. We assume that $H_{x,y} := \langle \delta_x \mid H\delta_y \rangle$ vanishes unless $|x-y| = 1/2$ and that H commutes with the translations $(S_y f)(x) := f(x-y)$ for all $y \in \mathbb{Z}$.

1. Identify $\mathcal{H} \cong \ell^2(\mathbb{Z}, \mathbb{C}^2)$ using $\Lambda_0 = \mathbb{Z} \sqcup (\frac{1}{2} + \mathbb{Z})$ and show that, under this identification, H corresponds to the operator on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ given by the matrix
$$H = \begin{pmatrix} 0 & a + bS_1 \\ \overline{a} + \overline{b}S_1^* & \end{pmatrix}$$
for some $a, b \in \mathbb{C}$.

2. Now use the identification $\Lambda_0 = \mathbb{Z} \sqcup (-\frac{1}{2} + \mathbb{Z})$ to identify $\mathcal{H} \cong \ell^2(\mathbb{Z}, \mathbb{C}^2)$. Show that H now has the matrix
$$H = \begin{pmatrix} 0 & aS_1^* + b \\ \overline{a}S_1 + \overline{b} & \end{pmatrix}.$$
This is interesting because this matrix and the one above have different indices for $|b| > |a|$.

Exercise 2. Let $\Lambda = \mathbb{Z}^2 \subseteq \mathbb{R}^2$ and let $\mathcal{H} = \ell^2(\Lambda)$. Let G be the group of all isometries of the plane $\mathbb{R}^2$ that map Λ to itself. Let $g \in G$ act on $\mathcal{H}$ by $(S_g f)(x) = f(g^{-1})(x)$ for all $x \in \Lambda$, $f \in \ell^2(\Lambda)$. Let H be a selfadjoint operator on $\mathcal{H}$ that commutes with this representation of G.

1. Show that G is the semidirect product of the group of translations S_x for $x \in \mathbb{Z}^2$ and the 8-element dihedral group of symmetries of the square.
2. Identify all the points $x \in \Lambda$ with $\|x\| \le \sqrt{2}$ and show that the action of the dihedral group restricted to this subset of Λ has exactly three distinct orbits.
3. Describe the most general form for H if $\langle \delta_x \mid H\delta_y \rangle$ vanishes for $\|x-y\| > \sqrt{2}$.
4. Assume now that $\langle \delta_x \mid H\delta_y \rangle = 0$ for $\|x-y\| \ne 1$. Compute the spectrum of H in this case. (Follow the sketch I gave for the case of graphene.)

In the following exercise, you may use that the Fredholm index of an operator is invariant under homotopies of Fredholm operators.

Exercise 3. Let $\hat{S} \in \mathbb{B}(\ell^2(\mathbb{N}))$ be the unilateral shift $\hat{S}f(x) = f(x-1)$. Let $\lambda \in \mathbb{C}$.

1. Assume $|\lambda| > 1$. Show that $\hat{S} - \lambda$ is invertible.
2. Assume $|\lambda| < 1$. Compute the kernels of the operators $\hat{S} - \lambda$ and $(\hat{S} - \lambda)^*$ for $|\lambda| \ne 1$ and prove that the image of $\hat{S} - \lambda$ is the entire orthogonal complement of the kernel of $(\hat{S} - \lambda)^*$. Deduce that $\hat{S} - \lambda$ is Fredholm.
3. Show that $\hat{S} - \lambda$ for $|\lambda| < 1$ and $|\lambda| > 1$ have different indices. Deduce that $\hat{S} - \lambda$ cannot be Fredholm for $|\lambda| = 1$.