Mathematisches Institut
Georg-August-Universität Göttingen
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Noncommutative geometry III
Winter semester 2024-2025

Exercise sheet 6.

Name

Exercise group (tutor's name)

Exercise	**1**	**2**	**3**	**Σ**
Points				

Deadline: **Friday, 29.11.2024, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let T, S_1, S_2 be operators on a Hilbert space $\mathcal{H}$ such that $TS_j - 1$ and $S_jT - 1$ are compact for $j = 1, 2$. Show that $S_1 - S_2$ is compact. Show that there is a parametrix S (that is, $ST - 1$ and $TS - 1$ are compact) that also satisfies $TST = T$ and $STS = S$. Give an example that shows that such an S need not be unique. (Hence, the Fredholm operators do *not* form an inverse semigroup.)

Exercise 2. Let A be a unital Banach algebra, $I \triangleleft A$ a closed 2-sided ideal.

1. Show that if $a \in A$ is close enough to 1_A then it is of the form $\exp(x)$ for some $x \in A$.
2. Show that if $a \in A$ is such that $[a] \in \left(A/I\right)^{\times}$ and is homotopic to $[1_A]$ then there exists $k \in I$ such that $a + k \in A^{\times}$.

Exercise 3. In this exercise, we introduce another equivalent point of view on Toeplitz operators. Denote by $\hat{S}$ the unilateral shift on $\ell^2(\mathbb{N})$ and $\mathcal{T} \subset \mathcal{B}(\ell^2(\mathbb{N}))$, the C^*-subalgebra generated by $\hat{S}$.

1. Show that a function in $L^2(\mathbb{T})$ extends holomorphically to the unit disk $\mathbb{D} \subset \mathbb{C}$ if and only if all its negative Fourier coefficients vanish. We denote by $H^2(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap L^2(\mathbb{T})$ this subspace. Show that this $H^2(\mathbb{D})$ is closed in $L^2(\mathbb{T})$.
   This gives $H^2(\mathbb{D})$ the structure of a Banach space for which $z \mapsto z^n, n \in \mathbb{N}$ forms an orthonormal basis.
2. Using the aforementioned basis, we get that a unitary isomorphism $F\colon \ell^2(\mathbb{N}) \to H^2(\mathbb{D})$. Compute the operator $F\hat{S}F^*$.
3. We write $P\colon L^2(\mathbb{T}) \to H^2(\mathbb{D})$ for the orthogonal projector onto $H^2(\mathbb{D})$. Show that F induces an isomorphism $\mathcal{T} \cong C^*(PM_f, f \in \mathcal{C}(\mathbb{T}))$.