Exercise sheet 7.

Name

Exercise group (tutor's name)

Exercise	1	2	3	Σ
Points				

Deadline: **Friday, 6.12.2024, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let A be a unital Banach algebra. Give $\mathbb{M}_2(A)$ the $\mathbb{Z}/2$-grading γ by conjugation with the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, that is, $\mathbb{M}_2(A)_+$ are the diagonal matrices and $\mathbb{M}_2(A)_-$ are the off-diagonal matrices. Prove that there is a homeomorphism

$$\{a \in A : a \text{ invertible}\} \cong \{a \in \mathbb{M}_2(A) : \gamma(a) = -a,\ a^2 = 1\}.$$

Exercise 2. Let (A, γ) be a $\mathbb{Z}/2$-graded C*-algebra. Show that $\{a \in A : \gamma(a) = -a,\ a^2 = 1\}$ deformation retracts onto the subset $\{a \in A : \gamma(a) = -a,\ a^2 = 1,\ a = a^*\}$, by following the steps below:

1. By polar decomposition, there is a unitary u so that $x = u|x|$ with $|x| = (x^*x)^{1/2}$. Show that $\gamma(u) = -u$ and $\gamma(|x|) = |x|$.
2. Use $x^{-1} = x$ to show that $x = u^*(u|x|^{-1}u^*)$ is another polar decomposition of x. Since the polar decomposition is unique, it follows that $u = u^*$, $|x| = u|x|^{-1}u^*$, so $u^2 = 1$.
3. Deduce $|x|^t = u^*|x|^{-t}u$ for all $t \in [0, 1]$.
4. Show that $t \mapsto u|x|^t$ for $t \in [0, 1]$ gives the required deformation retraction.

Exercise 3. Let $A = \mathrm{C}_0(X)$ for a locally compact Hausdorff space. Show that any conjugate-linear involutive automorphism α of A is of the form $\alpha(f)(x) = \overline{f(\tau(x))}$ for a unique involutive homeomorphism τ on X, that is, τ is a continuous map $\tau\colon X \to X$ with $\tau^2 = \mathrm{id}_X$. Conversely, these maps indeed define conjugate-linear involutive automorphisms of A. You may use in the proof that X is homeomorphic to the spectrum of A.

(It follows from this that any commutative real C*-algebra is of the form

$$\left\{ f \in \mathrm{C}_0(X) : f(x) = \overline{f(\tau(x))} \text{ for all } x \in X \right\}$$

for a unique pair (X, τ), where X is a locally compact space and τ is an involutive homeomorphism of X.)