Mathematisches Institut
Georg-August-Universität Göttingen
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Noncommutative geometry III
Winter semester 2024-2025

Exercise sheet 9.

Name

Exercise group (tutor's name)

Exercise	1	2	3	4	Σ
Points					

Deadline: **Friday, 20.12.2024, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Together with the Theorem saying that van Daele's K-theory is matrix-stable, the following exercise shows that conjugation by invertible elements acts identically on it.

Exercise 1. Let H be a functor on a category of Banach algebras that contains A and $\mathbb{M}_2(A)$. Assume that the maps $H(A) \to H(\mathbb{M}_2(A))$ induced by the two corner embeddings $A \rightrightarrows \mathbb{M}_2(A)$, $a \mapsto ae_{11}$ and $a \mapsto ae_{22}$ are both isomorphisms. Let $v \in A^\times$ be an invertible element. Prove that $H(\mathrm{Ad}_v) = \mathrm{id}_{H(A)}$. Hint: Use the automorphism of conjugation by the diagonal matrix $\left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

The following exercise is proven using the continuous functional calculus for selfadjoint elements in a C*-algebra. Using the holomorphic functional calculus for elements in a Banach algebra instead, one may prove a similar statement without selfadjointness, in any Banach algebra. If you are familiar with the holomorphic functional calculus, you may do this more general exercise. This may be used to prove a generalisation of the third exercise for a Banach algebra with a dense, increasing sequence of Banach subalgebras.

Exercise 2. *(This exercise says that an element that is approximately an odd selfadjoint involution is close to an odd selfadjoint involution.)* Let A be a $\mathbb{Z}/2$-graded unital C*-algebra with grading automorphism α. Let $\epsilon > 0$. Show that there is $\delta > 0$ such that the following holds: If $a \in A$ satisfies $\|\alpha(a) + a\| < \delta$, $\|a^* - a\| < \delta$, and $\|a^2 - 1\| < \delta$, then there is $b \in A$ with $\alpha(b) = -b$, $b = b^*$, $b^2 = 1$ and $\|a - b\| < \epsilon$. Hint: First find b_1 with $\|b_1 - a\| < \epsilon/2$ and $\alpha(b_1) = -b_1$, $b_1 = b_1^*$. Then let $b := \mathrm{sign}(b_1)$.

Exercise 3. *(This exercise shows that van Daele's K-theory is continuous for certain kinds of inductive limits, such as C*-algebraic inductive limits.)* Let A be a $\mathbb{Z}/2$-graded C*-algebra and let $A_n \subseteq A$ for $n \in \mathbb{N}$ be an increasing sequence of $\mathbb{Z}/2$-graded C*-subalgebras, such that $\bigcup_{n\in\mathbb{N}} A_n$ is dense in A. Assume that A is unital with $1 \in A_0$ and that $\mathcal{F}(A_0) \neq \emptyset$.

1. Show that for any $a \in \mathcal{F}(A)$ there are $n \in \mathbb{N}$ and $b \in \mathcal{F}(A_n)$ so that b and a are homotopic in $\mathcal{F}(A)$.
2. Show that if $a_0, a_1 \in \mathcal{F}(A_n)$ for some $n \in \mathbb{N}$ are homotopic in $\mathcal{F}(A)$, then there is $m \geq n$ so that they are homotopic already in $\mathcal{F}(A_m)$.

Hint: In both questions, use the homotopy between two elements in $\mathcal{F}(A)$ that are close enough. You may also use that on a C^*-algebra, $\mathcal{F}(A)$ deformation retracts onto its subset of self-adjoint elements.

Exercise 4. Apply matrix-stability of van Daele K-theory and the result of the previous exercise for

$$A_n := \hat{\mathbb{M}}_4\mathbb{M}_k(\mathbb{M}_n(B)^+), \qquad A := \hat{\mathbb{M}}_4\mathbb{M}_k(\mathbb{K}(\ell^2\mathbb{N}) \bar{\otimes} B)^+$$

for a $\mathbb{Z}/2$-graded C*-algebra B and $k \in \mathbb{N}_{\geq 1}$ to deduce that a corner embedding $B \to \mathbb{K}(\ell^2\mathbb{N}) \bar{\otimes} B$ induces an isomorphism $\mathrm{K}(B) \cong \mathrm{K}(\mathbb{K}(\ell^2\mathbb{N}) \bar{\otimes} B)$. Here $\bar{\otimes}$ denotes any C*-completion of the algebraic tensor product. Actually, this C*-completion is unique, but this is not relevant to the exercise. You may assume that $A_n \subseteq A$ is a C*-subalgebra as needed for the previous exercise. Here $\hat{\mathbb{M}}_4$ denotes the algebra of 4×4-matrices with the grading described in Exercise 1 of the previous sheet.